Proof strategy for Evaluation Theorem (Or: FTOC-2)

Prelude: How can one prove equality between any two entities, i.e., LHS=RHS?

One way: Start with LHS (or RHS) -- then manipulate it and transform it into the other.

E.g., Prove that $(x+1)^2 = x^2 + 2x + 1$

For the evaluation theorem, we must prove: $\int_a^b f(x)\,dx = F(b) - F(a)$

where $F(x)$ is a function that satisfies $\frac{dF}{dx} = f$.

Proof strategy:

(1) Start with RHS, which is $F(b) - F(a)$.

(2) Transform this into a summation involving n terms over the interval $[a, b]$.

E.g., $F(b) - F(a) = F(b) + \left[(p-p) + (q-q) + (r-r) + \ldots\right] - F(a)$

Select $p = F(x_{n-1}),\quad q = F(x_{n-2}),\quad r = F(x_{n-3}),\quad \ldots\;;$ with $b = x_n,\ a = x_0$

This gives: $F(b) - F(a) = F(x_n) - F(x_{n-1}) + F(x_{n-1}) - F(x_{n-2}) + \ldots + F(x_1) - F(x_0)$

$$\therefore\quad F(b) - F(a) = \sum_{k=1}^{n} \left[F(x_k) - F(x_{k-1})\right]$$

(Now we need to figure out a way to bring f into the above summation.)

(3) Use mean value theorem to relate differences in F to its derivative $\frac{dF}{dx}$.

(4) Use the condition that F is required to satisfy: $\frac{dF}{dx} = f$, to bring f into the picture. Take limits as $n \to \infty$, and we obtain the required LHS. Hence we have shown that LHS=RHS.

[**Recap:** What does the mean value theorem say?]

Proof strategy for Fundamental Theorem of Calculus - 1

Objective: Prove that $\frac{dg}{dx} = f(x)$ given $g(x) = \int_a^x f(t)\,dt$.

Proof strategy:

To prove LHS=RHS here we start with the LHS

(since the RHS doesn't offer much substance to work with).

(1) Here we have LHS=$\frac{dg}{dx}$. Expand $\frac{dg}{dx}$ using basic definition of derivative:

$$\frac{dg}{dx} = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}.$$

(2) Use the given defintion of $g(x)$ to figure out the numerator: $g(x+h) - g(x)$:

$$g(x+h) - g(x) = \int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt$$

(from additive property of D.I.'s)

$$\int_a^{x+h} f(t)\,dt = \int_a^x f(t)\,dt + \int_x^{x+h} f(t)\,dt$$

(3) Plug into step (1) to get: $\frac{dg}{dx} = \lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f(t)\,dt$.

We're almost done! All that remains is to convince ourselves that the RHS is indeed the same as $f(x)$.

There are different approaches. One easy strategy is to use a Riemann sum to compute the integral:

$$\frac{1}{h}\int_x^{x+h} f(t)\,dt \approx \frac{1}{h}\left[f(x) + f(x+\Delta x) + f(x+2\Delta x) + \ldots (n \text{ rectangles})\right]\Delta x \qquad \left\{\text{with } \Delta x = \frac{h}{n}\right\}$$

$$\therefore \lim_{h \to 0} \frac{1}{h}\int_x^{x+h} f(t)\,dt = \lim_{h \to 0}\left\{\lim_{n \to \infty}[f(x) + f(x+\Delta x) + f(x+2\Delta x) \ldots]\frac{\Delta x}{h}\right\} = [nf(x)]\frac{1}{n}$$

Examples of questions based on "guided" or "partial" proofs

The evaluation theorem (FTOC-2) says if $f(x)$ is continuous on the interval $[a, b]$ then:

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where $F(x)$ is any antiderivative of $f(x)$.

* This theorem can be proved by starting with $F(b) - F(a)$ and manipulating it through a series of steps that eventually yield the definite integral on the left.

Step 1:

We split the interval $[a, b]$ into n equal subdivisions, and label the points $x_0, x_1, x_2, \cdots, x_n$. This would require that $x_0 = a$ and $x_n = b$.

It can then be shown that $F(b) - F(a) = \sum_{i=1}^{n} [\, F(x_i) - F(x_{i-1})\,]$.

Verify that this equation is true when n=2 and when n=4.

Step 2:

The mean value theorem allows us to relate changes in function value, to the value of its derivative.

We want to apply the MVT to each term in the above sum. So, we notice that

$$\frac{F(x_i) - F(x_{i-1})}{x_i - x_{i-1}} = F'(x_i^*) \quad \text{OR} \quad F(x_i) - F(x_{i-1}) = F'(x_i^*)(x_i - x_{i-1})$$

which should hold for some x_i^* on the i^{th} interval.

The end result of applying this to the above sum is:

$F(b) - F(a) =$ **?**

Step 3:

We know that $F(x)$ is an antiderivative of $f(x)$. Thus: $F'(x_i^*) = f(x_i^*)$.

Plug this into the expression above, and we get:

$F(b) - F(a) =$ **?**

Step 4:

Notice that the R.H.S. now looks like a familiar Riemann sum. Let $n \to \infty$ and take the limit. By the definition of a definite integral, we now have the following result:

$F(b) - F(a) =$ **?**

The proof is complete!

The Fundamental Theorem of Calculus (FTOC-1) says if $f(x)$ is continuous on the interval $[a, b]$ then the area function $g(x) = \int_a^x f(t)\, dt$ has derivative equal to $f(x)$.

i.e., $g'(x) = f(x)$ for all x on $[a, b]$.

* To prove this theorem we must show that $g'(x) = f(x)$ [given how $g(x)$ is defined].
* Start with $g'(x)$ and manipulate it through a series of steps that eventually yield $f(x)$.

Step 1:

Using the basic definition of a derivative, express $g'(x)$ as the limit of a difference quotient (i.e., $\lim_{h \to 0}$ of stuff). This gives:

$g'(x) = \ ?$

Step 2:

Use the definition of $g(x)$ to rewrite the numerator in terms of definite integrals.

New numerator = ?

Simplify the numerator to a single integral by applying interval properties of definite integrals. This gives:

Numerator = ? Therefore, $g'(x) = \ ?$

Step 3:

We now convert the numerator from a definite integral to a Riemann sum. Split the interval $[x, x+h]$ into n equal subdivisions, and label the points $x_0, x_1, x_2, \cdots, x_n$. This would require that $x_0 = x$ and $x_n = x + h$. What is Δx?

$\Delta x = \ ?$ New numerator = ? (be sure to clarify what your x_i values are - e.g., $x_0 = x,\ x_i = x + i\Delta x,\ x_n = ?$) New numerator divided by h = ?

Step 4:

Plug into expression for $g'(x)$ that you found in Step 2, and take the limit $h \to 0$.

This gives:

$g'(x) \approx \ ?$

Next, take the limit as $n \to \infty$. [This is a trivial, formal step, since n has already canceled out, and we have the final result we want!] Thus, we get:

$g'(x) =$ **?**