Concept of Limit

* The limit of $f(x)$ as $x \to a$ is "sort of like" the function value at $x = a$, i.e., $f(a)$.

* In fact, we often find that for many values of a, the limit is equal to $f(a)$.

* But **...** this similarity is delusional!

> Conceptually, the limit of $f(x)$ as $x \to a$ is **unrelated** to $f(a)$.

* Best way to understand limits is to look at graph of $f(x)$.

* Recipe to find the limit as $x \to a$:

(1) "Get on" the graph somewhere near the point $x = a$.

(2) Head towards $x = a$, and note the y-value when you reach $x = a$.

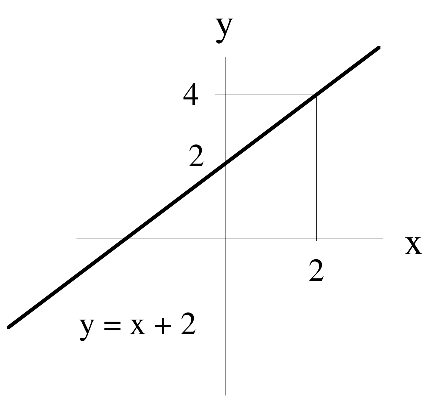
(3) Repeat steps (1) and (2) by approaching from the opposite side.

(4) If you get the same y-value, this is the limit.

Example1: $f(x) = x + 2$

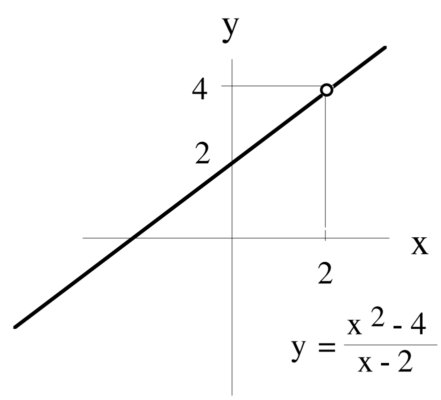
Example2: $f(x) = \dfrac{x^2 - 4}{x - 2}$

For both examples, find the limit of $f(x)$ as $x \to 2$.



At x=2, the function value and limit value are the same:

$\lim_{x \to 2} f(x) = 4 = f(2)$



At x=2, function value and limit value not the same:

$\lim_{x \to 2} f(x) = 4.$ But $f(2)$ = undefined.

Think: Now, compare function value and limit value at some other point, say, x=1.

Moral of the Story:

* The limit concept is inherently "neighborhood" based.
* You have to look at the function's behavior in the neighborhood of $x = a$.
* You must ignore the function value at the point $x = a$ itself.

Function value at $x = a$	**Limit value of $f(x)$ as x approaches a**
Only depends on $f(a)$.	Does not depend upon $f(a)$.
Does not depend upon neighboring values.	Only depends on neighboring values.
May exist even if $\lim_{x \to a} f(x)$ does not exist.	May exist even if $f(a)$ does not exist.

Left limit and right limit

* **Left limit**: Only look at graph to the left of the point of interest ($x = a$).
Ignore graph to the right.

"Get on" the graph & approach $x = a$ from the left.

The y-value when you reach $x = a$ is the left limit.

Basically we're saying, only consider the neighborhood to the left.

* **Right limit**: Only look at graph to the right of the point of interest ($x = a$).
Ignore graph to the left.

"Get on" the graph & approach $x = a$ from the right.

The y-value when you reach $x = a$ is the right limit.

Basically, only consider the neighborhood to the right.

* **Example**: In Example2 on the previous pages, consider the point x=2.

Left limit (LL): $\lim_{x \to 2^-} f(x) = 4$

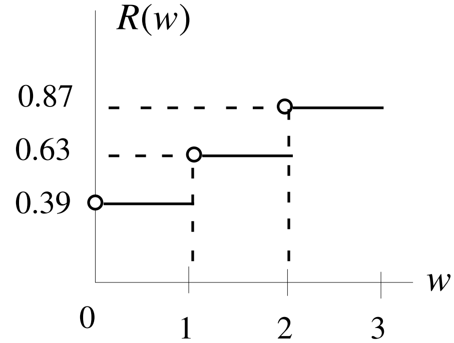
Right limit (RL): $\lim_{x \to 2^+} f(x) = 4$

Think: Are the LL and RL the same always? Why, or why not?

Example3: **(Postal rates)**

In 2006 the rate R of first class postage (in cents) as a function of the weight w (in ounces), was given by:

$$R(w)=\begin{cases}0.39, & \text{if } 0<w\le 1\\ 0.63, & \text{if } 1<w\le 2\\ 0.87, & \text{if } 2<w\le 3\end{cases}$$

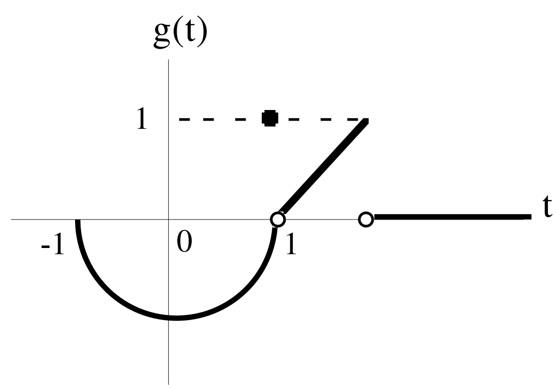


Find $\lim_{w\to 1} R(w)$, $\lim_{w\to 1^-} R(w)$, $\lim_{w\to 1^+} R(w)$.

Find $\lim_{w\to 1.5} R(w)$, $\lim_{w\to 1.5^-} R(w)$, $\lim_{w\to 1.5^+} R(w)$.

Example4: Consider the function $g(t)$ defined in the domain $-1\le t<\infty$:

$$g(t)=\begin{cases}-\sqrt{1-t^2}, & \text{if } -1\le t<+1\\ +1, & \text{if } t=1\\ t-1, & \text{if } 1<t\le 2\\ 0, & \text{if } t>2\end{cases}$$



Consider three separate points $t=a$, with a=0, a=1, a=2. For each point, answer the following questions:

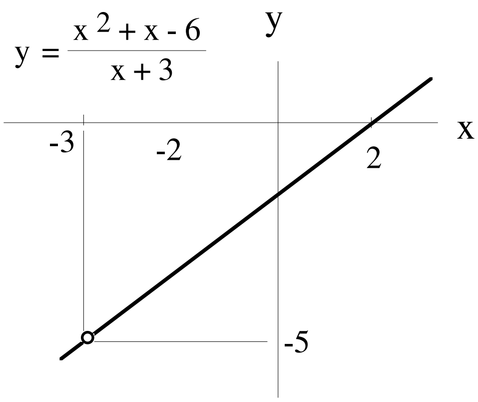
(1) What is $g(a)$?

(2) What is $\lim_{t\to a^-} g(t)$, $\lim_{t\to a^+} g(t)$, $\lim_{t\to a} g(t)$?

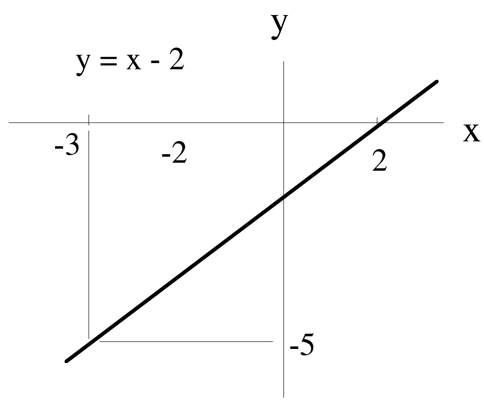
(3) Is the function continuous at $t=a$?

Example: **Find** $\lim_{x \to -3} \dfrac{x^2+x-6}{x+3}$.

Before algebra



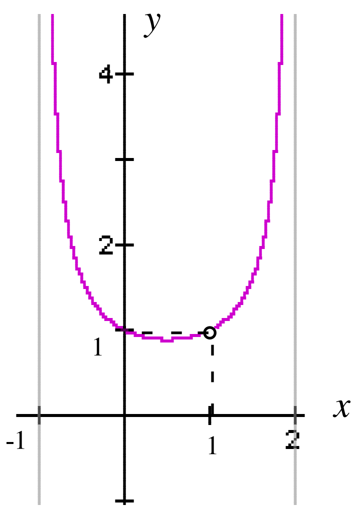
After algebra



Example: **Find** $\lim_{x \to 1} \left(1+\dfrac{1}{x-2}\right)\left(\dfrac{2}{1-x^2}\right)$.

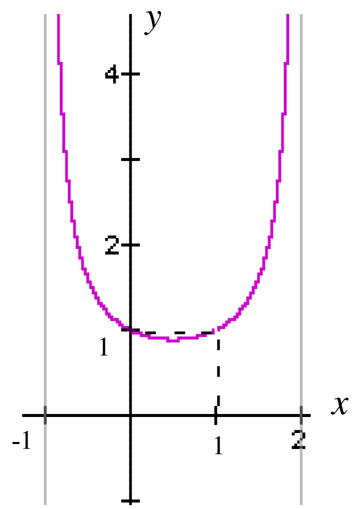
Before algebra

$$y=\left(1+\frac{1}{x-2}\right)\left(\frac{2}{1-x^2}\right)$$



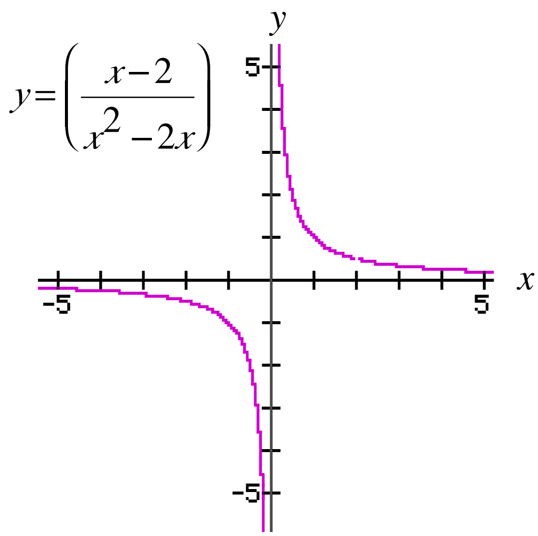
After algebra

$$y=\left[\frac{-2}{(x-2)\,(1+x)}\right]$$

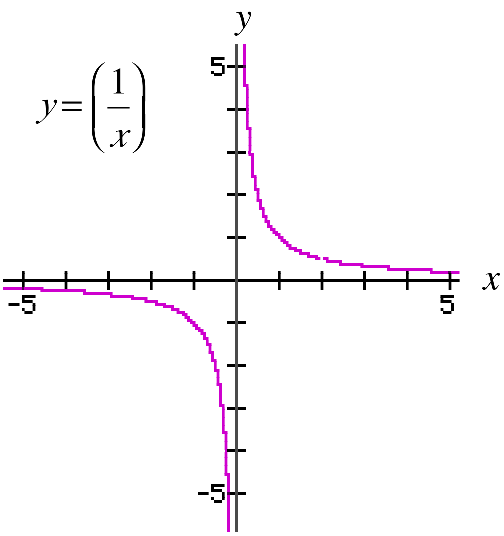


Example: Find $\lim\limits_{x \to 0} \dfrac{x-2}{x^2-2x}$.

Before algebra

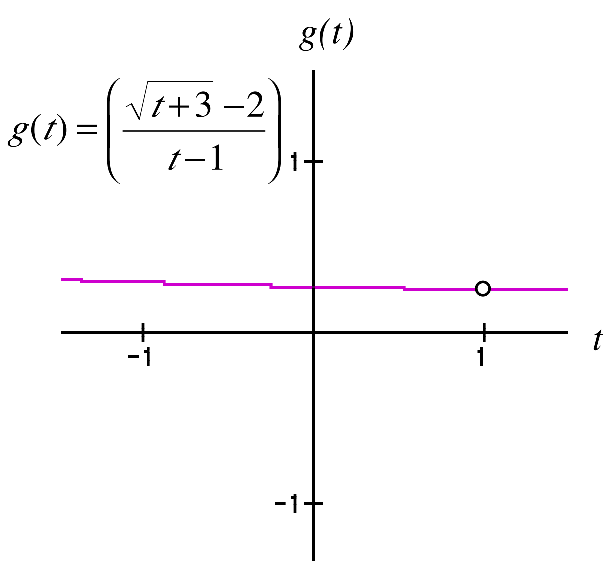


After algebra



Example: Find $\lim\limits_{t \to 1} \dfrac{\sqrt{t+3}-2}{t-1}$.

Before algebra



After algebra