The Chain Rule

Objective: To differentiate more general, complicated functions.

Key idea

* Treat complicated functions as composites assembled from simpler functions.

* Examples:

(1) $f(x) = \left(x^3 - 2x^2 + \frac{1}{x}\right)^{10} \quad \longleftrightarrow \quad f(u) = u^{10},\ u(x) = \left(x^3 - 2x^2 + \frac{1}{x}\right)$

(2) $g(x) = e^{\left(x^3 - 2x^2 + \frac{1}{x}\right)} \quad \longleftrightarrow \quad g(u) = e^u,\ u(x) = \left(x^3 - 2x^2 + \frac{1}{x}\right)$

(3) $r(t) = \ln\left(t^2 + 5t\right) \quad \longleftrightarrow \quad r(u) = \ln(u),\ u(t) = \left(t^2 + 5t\right)$

(4) $g(t) = \sqrt[5]{\ln\left(1 - t^5\right)} \quad \longleftrightarrow \quad g(u) = \sqrt[5]{u},\ u(v) = \ln(v),\ v(t) = \left(1 - t^5\right)$

* It is now important to distinguish between differentiation variables.

In example (1): $\frac{df}{dx}$ is not the same as $\frac{df}{du}$ or $\frac{du}{dx}$

In example (4): $\frac{dg}{dt}$ is different from $\frac{dg}{du}, \frac{du}{dv}$ and $\frac{dv}{dt}$

* Thus, the prime notation can become very misleading or confusing here.

Recall the general power rule ("baby" chain rule):

If your function has the form $f(x)=u^n$ then its derivative is

$$\frac{df}{dx} = n\,u^{n-1} \times \frac{du}{dx}$$

The full chain rule says:

If you write (or imagine) the function f(x) as a composite of the form f(u)•u(x),

then its derivative with respect to x is

$$\frac{df}{dx} = \frac{df}{du} \times \frac{du}{dx}$$

Recipe for applying the full chain rule to find df/dx:

$$\left\{ \text{e.g., let } f(x) = e^{\left(x^3 - 2x^2\right)} \right\}$$

Step 1: Simplify the function by writing it as composite of u(x).

$$\left\{ \text{e.g., let } f(u) = e^u, \quad u(x) = \left(x^3 - 2x^2\right) \right\}$$

Step 2: Differentiate f(u) with respect to u, and u(x) with respect to x, to get

$\frac{df}{du}$ and $\frac{du}{dx}$.

$$\left\{ \text{e.g., } \frac{df}{du} = e^u, \quad \frac{du}{dx} = \left(3x^2 - 4x\right) \right\}$$

Step 3: Get df/dx from chain rule: $\frac{df}{dx} = \frac{df}{du} \times \frac{du}{dx}$.

$$\left\{ \text{e.g., } \frac{df}{dx} = e^u \left(3x^2 - 4x\right) \right\}$$

Step 4: Replace "u" by original stuff.

$$\left\{ \text{e.g., } \frac{df}{dx} = e^{\left(x^3 - 2x^2\right)} \left[3x^2 - 4x\right] \right\}$$

Implicit Differentiation Preliminaries

What are implicit functions?

* Examples:

(A) $y^2 - x = 0$; (B) $x e^y - y = x^2 - 2$; (C) $2y + xy - 1 = 0$

* Key feature:

- y is not defined solely in terms of x.
- Function definition consists of mixture of x, y terms.
- Sometimes it is possible to solve for "y" and rewrite as y=f(x), but often it is not.

* Some questions to think about:

(1) How do you tell which variable is dependent and which independent?

E.g., think about the above 3 examples.

(2) Is it still a function? How can we tell?

(3) How can we graph such equations, even with a calculator?

Watch out for the variable of differentiation

* Suppose y is an implicit function of x: y = y(x).

* Any function created from y(x), can be differentiated with respect to x.

* Differentiate the following terms (or, function) as instructed:

(A) x^2 with respect to x.

(B) y^2 with respect to y.

(C) y^2 with respect to x.

(D) $\sqrt{y}$ with respect to y.

(E) $\sqrt{y}$ with respect to x.

(F) $x\,y^2$ with respect to x.

* Answers:

(A) 2x

(B) 2y

(C) $$\frac{dy^2}{dx} = \frac{dy^2}{dy}\,\frac{dy}{dx} = 2\,y\,\frac{dy}{dx}$$

(D) $$\frac{d\sqrt{y}}{dy} = \frac{dy^{1/2}}{dy} = \frac{1}{2}\,y^{-1/2}$$

(E) $$\frac{d\sqrt{y}}{dx} = \frac{d\sqrt{y}}{dy} \times \frac{dy}{dx} = \frac{1}{2}\,y^{-1/2}\,\frac{dy}{dx}$$

(F) $$\frac{d\,(xy^2)}{dx} = x\,\frac{dy^2}{dx} + y^2\,\frac{dx}{dx} = x\left[2\,y\,\frac{dy}{dx}\right] + y^2 = 2\,x\,y\,\frac{dy}{dx} + y^2$$