How to Prepare for Test 1

Preface: Many good students prepare poorly for math exams. In my view, the two most critically important ingredients for preparing well are: (1) extensive practice, and (2) constantly testing your understanding throughout the preparation process. In math it is very easy to delude yourself into thinking you understand something, or that you know how to solve a certain type of problem, even when you really don't. Nobody is immune from this – not even math professors! Thus, my advise to students is to get lots of practice and to keep testing your understanding along the way. What follows below are some specific directions you might take while getting your practice.

[1] Work through old quizzes, old exams, and key homework problems.

- best practice is to solve problems from scratch, without first looking at any solutions you may have
- try to add variations & create new problems as you gain comfort & insight

[2] Review your class notes. Pay special attention to:

- solution strategies & examples covered
- theoretical concepts that were indicated to be particularly important
- concepts that you had trouble with

[3] Rework the lab exercises.

[4] Rework key homework problems, particularly those you found difficult or didn't get right.

[5] Re-read sections of the textbook that deal with key concepts, as well as those you found difficult or unclear.
I also recommend the "Concept Check" and "True-False quiz" at the end of each chapter.

[6] Work through extra problems, beyond those assigned – especially those that have answers in the textbook. Remember to check the "Review" section at the end of each chapter in the book. There are also practice problems and exams on various websites linked to our class homepage.

[7] Seek help when you encounter trouble spots and to strengthen any areas that feel weak or murky.

Included topics & syllabus

The exam will be based on material from the following sections of the textbook:

Sections 1.5-1.6, 2.1-2.5.

Proficiency in Sections 1.1-1.3 topics is also expected, but no question will directly test this material.

Other key reminders

* Grading will be based on solution process and reasoning – NOT on right answers.

* Remember to bring a graphing calculator, and to use it for verifying your answers and for insights into solution strategies.

* Time-management is important when taking any test. The time-limit will be strictly enforced. Here are some pointers that may be helpful:

- Budget your time allotment for each problem at the very start.

  E.g., if there are 10 problems, and you have a total of 80 minutes, you can average upto 8 minutes per problem. To play it safe, cut that to 6 or 7 minutes allocated to each problem.

- Pick the "low-hanging" fruit first

  (i.e., start with the easy problems)

- If you're stuck, and not making progress for several minutes, move on to the next problem. You can return to the "stuck problem" at the end.

Concepts & Definitions

1. Functions. What is:
   (a) A function?
   (b) An independent variable; a dependent variable?
   (c) Domain; range of a function?
   (d) The graph of a function? the vertical line test?
   (e) A piecewise defined function?
   (f) The absolute value function $|x|$?
   (g) An increasing function; a decreasing function?
   (h) An odd function; an even function?
   (i) A linear, quadratic, polynomial, rational function? domains of these functions?
2. The composition of two functions $f(x)$ and $g(x)$?
3. The inverse $f^{-1}(x)$ of a function $f(x)$? How can we obtain their graphs from each other? How to algebraically find f^{-1} from f?
4. Exponential & log functions:
   - how are they related to each other – conceptually & graphically?
   - domain, range and key properties of their graphs
   - algebraic properties of log functions.
5. Limits of functions. Explain in words, and via geometry, the concept of:
   (a) Limit. What does $\lim_{x \to a} f(x) = L$ mean if I give you numbers in the place of a and L?
   (b) One-sided limits. What are they & how are they related to the "true" limit? e.g., only when left-limit & right-limit exist, and are equal does "true" limit exist.
   (c) Limits at infinity. How to find vertical & horizontal asymptotes of $f(x)$?

6. Continuity. What does it mean that a function $f(x)$:
    (a) Is continuous at a?
    (b) Is continuous from the right (or from the left) at a?
    (c) Is continuous on (a,b), $[a,b)$, $[a,b]$, $(-\infty,b]$, (a,∞), etc.
7. The tangent line to the graph of a function $f(x)$ at $x=a$. a secant line of the graph of $f(x)$.
8. The average velocity and the instantaneous velocity of an object whose position is recorded by $s(t)$.

Theorems

Must be able to explain what each of the following theorems say. Be sure to understand, distinguish and state the conditions (hypothesis) of each theorem and its conclusion. Be prepared to give examples for each theorem

* The Squeeze (Sandwich) Theorem.

* The Intermediate Value Theorem (IVT)

Other Theorems:

These are useful, but I won't ask you to state or explain what they mean!

* Limits Laws (Section 2.3): addition, subtraction, multiplication, division, multiplication by a constant, powers, roots. (Be careful about the division law! What extra conditions does it require?)

* Continuity Theorems (Sec. 2.4: p.120): addition, subtraction, multiplication, division, multiplication by a constant. (Be careful about the division law! What extra conditions does it require?)

* Special Continuity Theorems (p. 120-123, Theorems 5, 7, 8, 9): continuity statements about various types of functions, e.g., polynomials, rational, root, trigonometric, exponential and logarithmic functions.

Problem Solving Techniques

1. How do we find $\lim_{x\to a} f(x)$ when plugin fails (i.e., the limit laws fail)?

   (a) If $f(x) = \dfrac{g(x)}{h(x)}$ and plugging in a yields $\dfrac{0}{0}$, try factoring polynomials, putting fractions to a common denominator, and rationalizing expressions with square roots. The idea is to end up with $(x-a)$ both in numerator and denominator, cancel it, and then again attempt to apply the Limit Laws.

   (b) If $f(x)$ is a piecewise-defined function (i.e., given by different formulas on different intervals), try to find the left-hand and the right-hand limits separately, and then compare them to see if they are equal (or if they exist, for that matter).

   (c) If $f(x)$ is given by a formula involving absolute values, again proceed by finding and comparing the two one-sided limits.

2. How do we determine if a function $f(x)$ is continuous at $x = a$?
   By definition of continuity, there are 3 things to check:

   (a) Find $f(a)$. If f is not defined at a, then the function is not continuous at a.

   (b) Find $\lim_{x\to a} f(x)$ by following either limits laws or the techniques suggested above. (If it doesn't exist, then the function has no chance of being continuous at a.)

   (c) If the above two steps yield two finite numbers, are they are equal: $\lim_{x\to a} f(x) = f(a)$? If yes, the function is continuous at a.

3. How to tell if a function is continuous at a without using the definition of continuity? We use the continuity theorems if applicable. For example, the function $f(x) = \dfrac{\cos(x)}{x-3} - 6x^3$ is continuous at 2 because all comprising functions (rational, trigonometric and polynomial) are continuous at 2. However, the function is not continuous at 3. (Why?)

4. How to find the vertical and horizontal asymptotes of any $f(x)$?

   (a) To find vertical asymptotes we must locate the x-values where $f(x)$

approaches $\pm\infty$. In mathematical terms, we must find all points $x = a$ where $\lim_{x \to a} f(x) = +\infty$ or $-\infty$.

If $f(x)$ is of rational form (ratio of polynomials), these x-values can often be found by determining where its denominator=0. However, be sure to check that the numerator is NOT 0 at the same point. (Can you tell why that is important?)

If $f(x) = \ln(\text{stuff})$ then there is a vertical asymptote where stuff $= 0$.

(b) To find the horizontal asymptotes we must evaluate the limits:

$\lim_{x \to +\infty} f(x)$ and $\lim_{x \to -\infty} f(x)$.

Some useful facts of algebra

1. Manipulations with fractions.

   (a) Splitting fractions: $\dfrac{a+b}{c} = \dfrac{a}{c} + \dfrac{b}{c}$; $\quad \dfrac{ab}{cd} = \dfrac{a}{c} \cdot \dfrac{b}{d} = \dfrac{a}{d} \cdot \dfrac{b}{c} = \dfrac{a}{cd} \cdot b$, etc.

   (b) Wrong formula: $\dfrac{a}{b+c} \neq \dfrac{a}{b} + \dfrac{a}{c}$.

   (c) Putting fractions under a common denominator: $\dfrac{a}{b} + \dfrac{c}{d} = \dfrac{ad+bc}{bd}$.

   (d) When denominators have something in common: $\dfrac{a}{be} + \dfrac{c}{de} = \dfrac{ad+bc}{bde}$.

   (e) Fractions over fractions: $\dfrac{\frac{a}{b}}{\frac{c}{d}} = \dfrac{a}{b} \div \dfrac{c}{d} = \dfrac{ad}{bc}$; $\quad \dfrac{\frac{a}{b}}{c} = \dfrac{a}{bc}$; $\quad \dfrac{a}{\frac{c}{d}} = \dfrac{ad}{c}$.

2. Manipulations with exponentials.

$$a^b a^c = a^{b+c}; \quad \frac{a^b}{a^c} = a^{b-c}; \quad \left(a^b\right)^c = a^{bc}; \quad a^{\frac{b}{c}} = \sqrt[c]{a^b} = \left[\sqrt[c]{a}\,\right]^b$$

3. Rationalizing formula: $\sqrt{a} - \sqrt{b} = (\sqrt{a} - \sqrt{b}) \left[\dfrac{\sqrt{a}+\sqrt{b}}{\sqrt{a}+\sqrt{b}}\right] = \dfrac{a-b}{\sqrt{a}+\sqrt{b}}$

4. Absolute value items.

   $|x| < a$ means the same as $-a < x < a$.

   $|x| > a$ means the same as $x > a$ and $x < -a$.
   (You could say $-a > x > a$, but this can be confusing.)

   E.g., $\dfrac{x}{|x^2-6|} = \begin{cases} \frac{x}{x^2-6}, & \text{if } x^2 - 6 > 0 \\ -\frac{x}{x^2-6}, & \text{if } x^2 - 6 < 0 \end{cases}$

Rough distribution of questions on this test

- 1-2 questions: Define, state, explain, or illustrate some key concept or theorem.
- 1 question involving the use of log properties.
- 1 question involving inverse of a function.
- 2-3 questions: Find limits using algebra.
- 1-2 questions involving continuity: Prove that some function is continuous (or discontinuous); find points of discontinuity; determine whether continuous at specified point (including left/right continuity).
- 1 question involving finding asymptotes and/or limits at infinity.
- 1 question involving sketching and/or working with graphs: May include log, exponential, or piecewise functions; work may include finding limits, intervals of continuity, graph of inverse.
- Other possible questions: Anything (close to) assigned homework exercises, or self-study topics.