Student name:

MATH 180: Calculus A | **Final exam**
Spring 2022 | **May 19, 2022**

Instructions:

- This is a regular "closed-book" test, and is to be taken without the use of notes, books, or other reference materials. Collaboration or group work is not permitted.
- Cell-phone usage in any form is prohibited for the entire duration of the test. This also applies to any restroom breaks taken during the test.
- Answer all questions on separate paper (not on this sheet!).
- Solve all problems using algebra, except if specifically indicated otherwise. Show all solution steps, give reasons, and simplify your answer to receive full credit.
- The time limit for taking this test is 2 hours from the scheduled start time.
- This test contains questions numbered 1-7. It adds up to 50 points.

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(1) [4 pts.] Give a mathematically precise definition of:

(a) The derivative of a function.

(b) The definite integral $\displaystyle\int_a^b f(x)\,dx$.

(2) [4 pts.] Find the most general form of f, given $f' = 3e^x + \dfrac{1}{2x} + \sqrt{x}$.

(3) [4 pts.] Give brief answers to each of the following as instructed:

(a) Water is flowing out of a pipe at the rate of $g'(t)$ gallons per minute. Interpret what $\displaystyle\int_2^5 g'(t)dt = 68$ means in this context, and give its units.

(b) Suppose A is a differentiable function that represents the amount of a chemical (in mg.) present x minutes after the start of a chemical reaction. What are the units of $A'(x)$? What is the meaning of $A'(3) = -4$?

(4) [5 pts.$\times$ 2] Evaluate the following limits, showing all steps and reasons:

(a) $\displaystyle\lim_{t\to 3}\left(\frac{2}{t-3} - \frac{12}{t^2-9}\right)$

(b) $\displaystyle\lim_{x\to -\infty}\frac{1-4e^x}{1+2e^x}$ and $\displaystyle\lim_{x\to\infty}\frac{1-4e^x}{1+2e^x}$

(5) [5 pts.$\times$3] Differentiate each of the following with respect to x and simplify:

(a) $f(x) = |x|\sin(x)$

(b) $h(x) = \dfrac{x^4 - 3x + \sqrt{x}\sin x}{\sqrt{x}}$

(c) $y = (1+2x)^{1/x}$

(6) [5 pts.] Consider the function $g(x) = (x-2)^4(x+3)$. Show correct calculus and algebra steps for each of the following:

(a) Find all the local minimum and maximum values of g.

(b) Find the absolute minimum and maximum values of g on the interval $-3 \leq x \leq 5$.

(7) [8 pts.] All the following statements are false. For each, give an example (in the form of a clearly labeled graph, or an algebraic equation) showing the statement is false.

(a) If a function f is continuous on the interval $(0,4)$ then f' exists for all x in that interval.

(b) If $f(x) \geq g(x)$ for all x in $(0,4)$, then $f'(x) \geq g'(x)$ for all x in that interval.

(c) If $f'(x) \geq g'(x)$ for all x in $(0,4)$, then $f(x) \geq g(x)$ for all x in that interval.

(d) If $\int_0^4 f(x)dx \geq \int_0^4 g(x)dx$, then $f(x) \geq g(x)$ for all x in $[0,4]$.

End of test

Calculus A: Spring 2022: Final Exam Solutions

[1]
(a) The derivative of a function $f(x)$ at $x=a$ is defined by

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

OR

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

If this limit exists, then f has a derivative at $x=a$

(b) The definite integral of a function f on the interval $[a,b]$ is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x, \text{ where } \Delta x = \frac{b-a}{n}, \quad x_i = a + i\Delta x$$

x_i^* is any point in $[x_{i-1}, x_i]$

If this limit exists, then f is integrable on $[a,b]$.

[2] Given: $f' = 3e^x + \frac{1}{2x} + \sqrt{x}$

Taking antiderivatives term-by-term, we get

$$\boxed{f = 3e^x + \frac{1}{2}\ln|x| + \frac{2}{3}x^{3/2} + C}$$

C = arbitrary constant

[3] (a) $\int_2^5 g'(t)\,dt = 68$ means: Between minute 2 and minute 5 the net outflow of water from the pipe is 68 gallons.

(b) $A(x)$ = amount of a chemical in mg. x minutes after start of reaction

Units of $A'(x) = \frac{\text{mg.}}{\text{minute}}$

Meaning of $A'(3) = -4$: Three minutes after the start of the reaction, the rate of change in the amount of chemical present (with respect to time) is -4 mg/minute. Thus, the amount present is decreasing at the rate of 4 gm per minute.

[4]
(a) $$\lim_{t\to 3}\left(\frac{2}{t-3} - \frac{12}{t^2-9}\right) = \lim_{t\to 3}\left(\frac{2(t+3)}{(t-3)(t+3)} - \frac{12}{(t-3)(t+3)}\right)$$

$$= \lim_{t\to 3} \frac{2t+6-12}{(t-3)(t+3)} = \lim_{t\to 3} \frac{2\cancel{(t-3)}}{\cancel{(t-3)}(t+3)} = \frac{2}{6}$$

Answer: $\boxed{\frac{1}{3}}$

[4] (b) $\lim_{x\to-\infty} \frac{1-4e^x}{1+2e^x}$. Since $\lim_{x\to-\infty} e^x = 0$, we can write

this as $\frac{\lim_{x\to-\infty}(1-4e^x)}{\lim_{x\to-\infty}(1+2e^x)} = \frac{1-4(0)}{1+2(0)} \boxed{=1}$

And $\lim_{x\to\infty} \frac{1-4e^x}{1+2e^x}$ gives, upon plugin, $\frac{1-\infty}{1+\infty} \sim \frac{-\infty}{\infty}$ indeterminate

Apply L.H. rule: $\lim_{x\to\infty} \frac{1-4e^x}{1+2e^x} = \lim_{x\to\infty} \frac{-4e^x}{2e^x} = \lim_{x\to\infty} -\frac{4}{2} \boxed{=-2}$

Answers: $\boxed{\lim_{x\to-\infty} \frac{1-4e^x}{1+2e^x} = 1 \text{ and } \lim_{x\to\infty} \frac{1-4e^x}{1+2e^x} = -2}$

[5] (a) $f(x) = |x| \sin(x) = \begin{cases} x\cdot\sin(x), & \text{if } x \geq 0 \\ -x\cdot\sin(x), & \text{if } x < 0 \end{cases}$

$$\frac{df}{dx} = \begin{cases} x\cdot\cos(x) + \sin(x), & \text{if } x > 0 \\ -x\cdot\cos(x) - \sin(x), & \text{if } x < 0 \end{cases}$$

$$\left.\begin{array}{l} \text{As } x \to 0^+,\ \frac{df}{dx} \to 0+0 = 0 \\ \text{As } x \to 0^-,\ \frac{df}{dx} \to -0-0 = 0 \end{array}\right\} \Rightarrow \text{At } x=0,\ \frac{df}{dx} = 0$$

Answer: $\boxed{\frac{df}{dx} = \begin{cases} x\cdot\cos(x) + \sin(x), & \text{if } x \geq 0 \\ -x\cdot\cos(x) - \sin(x), & \text{if } x < 0 \end{cases}}$

(b) $h(x) = \frac{x^4 - 3x + \sqrt{x}\cdot\sin x}{\sqrt{x}} = x^{7/2} - 3x^{1/2} + \sin(x)$

Thus, $\frac{dh}{dx} = \frac{7}{2}x^{5/2} - 3\cdot\frac{x^{-1/2}}{2} + \cos(x)$

Answer: $\boxed{\frac{dh}{dx} = \frac{7}{2}x^2\sqrt{x} - \frac{3}{2\sqrt{x}} + \cos(x)}$

(c) $y = (1+2x)^{1/x}$.

Apply ln on both sides and differentiate implicitly.

All primes denote derivatives with respect to x.

$$\ln y = \ln(1+2x)^{1/x} = \frac{\ln(1+2x)}{x}$$

$$\Rightarrow \frac{1}{y}y' = \frac{x\cdot\left(\frac{1}{1+2x}\right)\cdot(2) - \ln(1+2x)}{x^2} = \frac{2x - (1+2x)\ln(1+2x)}{(1+2x)x^2}$$

$$\therefore y' = (1+2x)^{1/x}\frac{[2x - (1+2x)\ln(1+2x)]}{(1+2x)x^2}$$

$$\boxed{y' = \frac{(1+2x)^{(1/x - 1)}}{x^2}[2x - (1+2x)\ln(1+2x)]}$$

[6] Given $g(x) = (x-2)^4(x+3)$

(a) For local extremes, I'll find critical points and use a sign chart of $g'(x)$.

$$g'(x) = 4(x-2)^3(x+3) + (x-2)^4 = (x-2)^3(5x+10)$$

Thus the critical points, corresponding to $g'=0$, are: $x=2, x=-2$

Sign chart of $g'(x)$:

	$x<-2$	$-2<x<2$	$x>2$
$g'(x)$	+	−	+

$g'(-100) > 0$

$g'(0) = (-2)^3(10) < 0$

$g'(100) > 0$

Local minimum: $g(2) = 0$

Local maximum: $g(-2) = (-4)^4 = 256$

(b) For absolute extremes on $[-3, 5]$:

Plug in end points and c.p.'s into $g(x)$ and find values.

x	-3	-2	2	5
$g(x)$	0	256	0	648

Abs. minimum: $g(-3) = g(2) = 0$

Abs. maximum: $g(5) = 648$

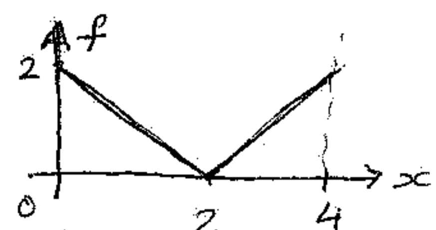
Answers: The local minimums are: $g(2) = 0$

The local maximums are: $g(-2) = 256$

The abs. minimum on $[-3,5]$ is: $g(-3) = g(2) = 0$

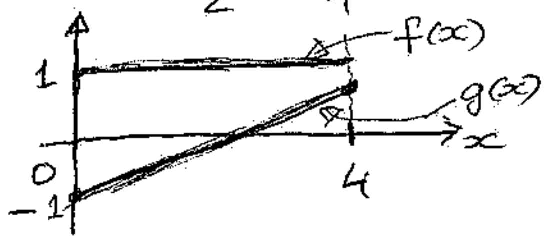
The abs. maximum on $[-3,5]$ is: $g(5) = 648$

[7] (a)



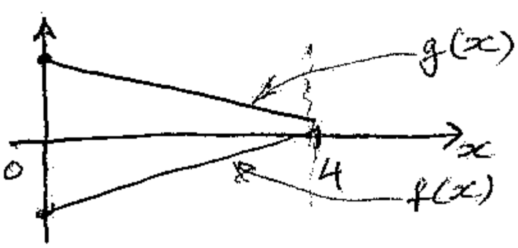
f shown here is continuous on $[0,4]$ but f' is undefined at $x=2$, because of the sharp corner.

(b)



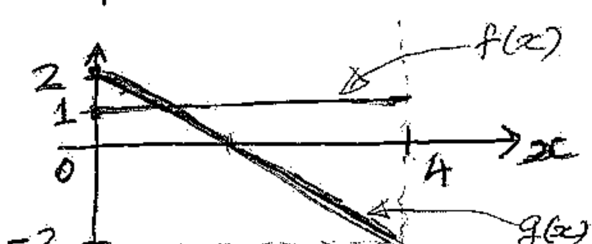
The graph shows a constant $f(x)$ and a linear $g(x)$ with positive slope. Thus, $f'(x) = 0$ and $g'(x) > 0$. Thus, $f(x) \geq g(x)$ on $(0,4)$ but $f' < g'$

(c)



In this graph, $f(x)$ has positive slope, and $g(x)$ has negative slope. Thus $f' > g'$, but $f < g$ on $(0,4)$.

(d)



Here $\int_0^4 f(x)\,dx = (1)(4) = 4$, and

$\int_0^4 g(x)\,dx = 0$ (or, my graph makes it look negative)

Thus $\int_0^4 f < \int_0^4 g$. But $f(x) \leq g(x)$

Grading Notes

[1] (a) = 2 points, (b) = 2 points
For each: −0.5 pt if limit missing. No other partial credit.

[2] 1 pt. each for correct integral of 3 terms. 0.5 pt = show minimal steps/reasons
0.5 pt = show constant of integration.

[3] (a) = 2 points, (b) = 2 points
(a) 0.5 pt = know it is from minute 2 to 5; 0.5 pt = correct unit of $\int_2^5 g'(t)\,dt$;
1 pt = correct rest of interpretation
(b) 1 pt = correct units of A'; 1 pt = interpretation of $A'(3) = -4$.

[4] (a) ~~1~~ pt = attempt to common denom; 2 pt = do common denom & simplify,
1 pt = factor and cancel $(t-3)$; 1 pt = plug in $t=3$ and get answer
(b) 2 points for $x \to -\infty$ part; 3 pt for $x \to \infty$ part
For $x \to -\infty$: 1+1 pt for answer + reason
For $x \to \infty$: 0.5 pt = check indet.; 2 pt = apply L·H. & simplify;
0.5 pt = answer

[5] (a) 1 pt = rewrite as correct piecewise fn with correct domain
2 pt = find correct derivative of both pieces
1 pt = specify correct domain boundaries on derivative
1 pt = justify/determine value of derivative at $x=0$.
(b) 1 pt = simplify & reduce to 3 terms;
1 pt each for derivative of 3 terms; 1 pt = write final answer in reasonable form
For QR based solution: 1 pt = know/show correct QR formula;
2 pt = get correct derivatives in numerator; 2 pt = simplify to reasonable form.
(c) 1 pt = apply ln and simplify R.H.S; 0.5 pt = correctly differentiate L.H.S.
2.5 pt = complete/correct derivative of R.H.S.; 1 pt = multiply by y and reverse sub.

[6] 2 pt = correct derivative of g, and algebraic work leading to 2 C.P.s
2 pt = correct sign analysis or chart showing +/− intervals
For (a): 1 pt = state correct func values of g at minimum/maximum
For (b): 1 pt = correct x-values in a table; answers carry no additional/separate credit

[7] (a) = (b) = (c) = (d) = 2 points