Student name:

MATH 180: Calculus A — **Test 3**

Fall 2019 — **December 3, 2019**

Instructions:

- This is a regular "closed-book" test, and is to be taken without the use of notes, books, or other reference materials. Collaboration or group work is not permitted.
- Cell-phone usage in any form is prohibited for the entire duration of the test. This also applies to any restroom breaks taken during the test.
- Answer all questions on separate paper (not on this sheet!).
- Solve all problems using algebra, except if specifically indicated otherwise. Show all solution steps, give reasons, and simplify your answer to receive full credit.
- The time limit for taking this test is 80 minutes from the scheduled start time.
- This test adds up to 50 points. It contains questions numbered (1) through (8).

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(1) [4 pts.] For each question below, sketch the graph of a function $f(x)$ that has the indicated property or, if that is impossible, explain why. As always axes must be completely & clearly labeled for credit.

(a) f has critical points at $x = 0$ and $x = 3$, is differentiable at those points, but has no local maximum or minumum anywhere.

(b) Same as question (a), except this time $f(x)$ is NOT differentiable at the critical points $x = 0$ and $x = 3$.

(2) [6 pts.] Find dy/dx for each the following:

(a) $y = \ln(\ln x)$ (b) $y = (\ln x)^x$

Hint: Did you use (a) for doing part of (b)?

(3) [6 pts.] (a) Use a Riemann sum with four rectangles and right endpoints to find the approximate value of $\displaystyle\int_0^2 xe^{-x}dx$.

(b) Generalize the above result to n rectangles. Your final result must be written in summation notation, with no unknowns in it except n. Show steps and reasons.

(4) [6 pts.] (a) Give a mathematically precise definition of the definite integral $\int_a^b f(x)dx$.

(b) Give an example illustrating the difference between a definite integral and an antiderivative, or indefinite integral. Be sure to explain the difference you've shown.

(5) [6 pts] Let $f(x) = (x + 3)^4(x - 2)^3$. Find all the local and absolute extreme values of f on the interval $[-4, 3]$. Credit for correct calculus and algebra steps only – not for answers!

(6) [6 pts.] According to recent data from the NOAA (a U.S. government agency), the global mean sea level has been rising at an accelerating rate over the past few decades. Let the function $f(x)$ represent the annual rate of change in mean sea level (in mm per year), with x representing the number of years since January 1, 2000.

(a) Describe what $\displaystyle\int_0^{15} f(x)dx$ represents in this application and give its units.

(b) A simple model that roughly approximates the real data is given by

$f(x) = 3.6 + 0.2\sqrt{x} - 0.1\cos x$. Compute $\displaystyle\int_0^{15} f(x)dx$ using the Evaluation Theorem, and interpret what your results mean in the context of this application.

(7) [6 pts] Find dy/dx, given: $y = \displaystyle\int_x^{e^x} \sqrt{t^4+3}\,dt$

(8) [10 pts.] For each question below, estimate a precise numerical value for the indicated quantity or, if that is impossible, explain why. Show steps/reasons. In all cases, assume f is a differentiable function on the interval $[-3,3]$. NOTE that all questions are independent, and are grouped together here only for convenience.

(a) Suppose $f(x) \geq 8$ for all x in $[-3,3]$. What is the minimum possible value of

$$\int_{-3}^{3} f(x)dx.$$

(b) Suppose $f(x) \geq 8$ for all x in $[-3,3]$. What is the minimum possible value of $f'(x)$ on that interval.

(c) Given $f(-3) = 8$, and $0 \leq f'(x) \leq 1$ for all x in $[-3,3]$, estimate the maximum possible value of $f(3)$.

(d) f has absolute maximum at $x = 1$, with $f(1) = 5$. Estimate the value of $f'(1)$.

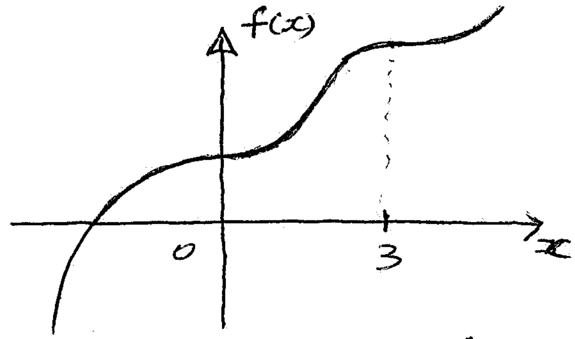
(e) Given $f(2) = f'(2) = -0.5$, estimate the value of $f(1.8)$ using linear approximation.

End of test

Calculus A: Fall 2019: Test 3 Solutions

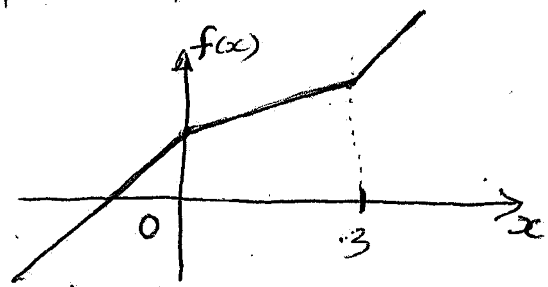
[1] (a) critical points: $x=0,\ x=3$
Differentiable at C.P.
No local minimum/maximum

One example of such a function is shown. $f'(0)=f'(3)=0$



(b) Same requirements as (a), except f is <u>not</u> differentiable at critical points.

The function shown has critical points at $x=0, x=3$, but is not differentiable at those points due to sharp corner.



[2] (a) $y=\ln(\ln x) \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{\ln x}\cdot\dfrac{d}{dx}(\ln x) = \dfrac{1}{\ln x}\cdot\left(\dfrac{1}{x}\right)$

$$\therefore \boxed{\frac{dy}{dx} = \frac{1}{x\cdot \ln x}}$$

(b) $y=(\ln x)^x$

Take ln on both sides: $\ln y = \ln\left[(\ln x)^x\right] = x\cdot\ln(\ln x)$

Differentiate both sides with respect to x:

$$\frac{1}{y}\cdot\frac{dy}{dx} = \ln(\ln x) + x\cdot\frac{d}{dx}\left[\ln(\ln x)\right]$$

From part (a), $=\dfrac{1}{x\cdot\ln x}$

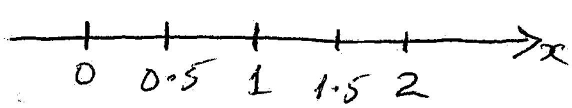
$$= \ln(\ln x) + \frac{\cancel{x}}{\cancel{x}\cdot\ln x}$$

$$\therefore \frac{dy}{dx} = y\left[\ln(\ln x) + \frac{1}{\ln x}\right]$$

$$\boxed{\frac{dy}{dx} = (\ln x)^x\left[\ln(\ln x) + \frac{1}{\ln x}\right]}$$

[3] (a) Approximate $\int_0^2 x\cdot e^{-x}\,dx$ with 4 rectangles & right end points

Let $f(x) = x\cdot e^{-x}$. With 4 rectangles $\Delta x = \dfrac{2-0}{4} = \dfrac{1}{2}$



x	$f(x)$
0.5	0.3033
1	0.3679
1.5	0.3347
2	0.2707

$$R_4 = [f(x_1)+f(x_2)+f(x_3)+f(x_4)]\,\Delta x$$
$$= [0.3033 + 0.3679 + 0.3347 + 0.2707]\left(\tfrac{1}{2}\right)$$

$$\boxed{R_4 = 0.6383}$$

[3] (b) with n rectangles we have: $\Delta x = \frac{2-0}{n} = \frac{2}{n}$

$x_i = 0 + i\frac{2}{n} = \frac{2i}{n}$. $\quad f(x_i) = x_i e^{-x_i} = \left(\frac{2i}{n}\right) e^{-\frac{2i}{n}}$

$$R_n = \sum_{i=1}^{n} \frac{2i}{n} e^{\frac{-2i}{n}} \cdot \frac{2}{n} = \sum_{i=1}^{n} \frac{4}{n^2} i e^{-2i/n}$$

Answer: $\boxed{R_n = \frac{4}{n^2} \sum_{i=1}^{n} i \cdot e^{-2i/n}}$

[4] If $f(x)$ is a function defined on $[a,b]$, the definite integral is

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

where $\Delta x = \frac{b-a}{n}$, $x_i = a + i\Delta x$, x_i^* is any point in $[x_{i-1}, x_i]$

If this limit exists, then it is the definite integral.

(b) Consider $f(x) = x$ on the interval $[0,1]$

Then, the definite integral $\int_0^1 x\,dx = \frac{1}{2}$, which can be shown by appling the definition or by using Evaluation Thm.

The antiderivative is $\int x\,dx = \frac{x^2}{2} + c$

An antiderivative is any function whose derivative is the given $f(x)$.

[5] Given: $f(x) = (x+3)^4(x-2)^3$

First, I will find the critical points of f.

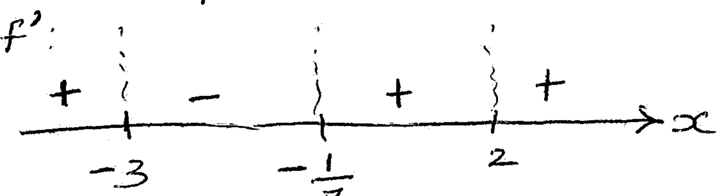
$$f'(x) = 4(x+3)^3(x-2)^3 + 3(x+3)^4(x-2)^2 = 0$$

$$\Rightarrow (x+3)^3(x-2)^2[4x-8+3x+9] = 0$$

$$\Rightarrow (x+3)^3(x-2)^2(7x+1) = 0$$

There are 3 critical points: $x=-3$, $x=2$, $x=-1/7$

Sign chart of f':



Thus, there is a local maximum @ $(-3, f(-3)) = (-3, 0)$

There is a local minimum @ $(-\frac{1}{7}, f(-\frac{1}{7})) = (-\frac{1}{7}, -655.7)$

For absolute extremes, let's look at f values at C.P. & end points

x	-4	-3	$-1/7$	3	2
$f(x)$	-216	0	-655.7	1296	0

The absolute minimum is: $(-\frac{1}{7}, -655.7)$

absolute maximum is: $(3, 1296)$

[6] $f(x)$ = annual rate of change in mean sea level (in mm/year)

(a) $\int_0^{15} f(x)\,dx$ = Net change in the mean sea level between the years 2000 and 2015.

Its units would be m.m.

(b) $f(x) = 3.6 + 0.2\sqrt{x} - 0.1\cos x \Rightarrow \int_0^{15} f(x)\,dx = \left[3.6x + \frac{0.2\,x^{3/2}}{3/2} - 0.1\sin x\right]_0^{15}$

$$= \left[3.6x + \frac{0.4}{3}x^{3/2} - 0.1\sin x\right]_0^{15}$$

$$= [54 + 7.746 - 0.065] - [0]$$

$$= 61.681 \text{ mm.}$$

Interpretation: Between the year 2000 and 2015 there has been a net rise in mean sea level by 61.681 mm.

[7] Given: $y = \int_x^{e^x} \sqrt{t^4+3}\,dt$. Since $\sqrt{t^4+3}$ is continuous for all t, the fundamental theorem of calculus is applicable here.

Using the properties of definite integrals, we can write

$$y = \int_x^1 \sqrt{t^4+3}\,dt + \int_1^{e^x} \sqrt{t^4+3}\,dt = -\int_1^x + \int_1^{e^x}$$

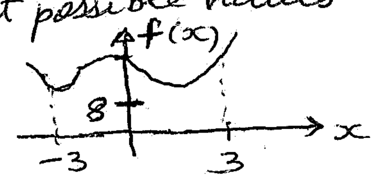
By the F.T.C.,

$$\frac{dy}{dx} = -\frac{d}{dx}\int_1^x \sqrt{t^4+3}\,dt + \frac{d}{dx}\int_1^{e^x} \sqrt{t^4+3}\,dt = -\sqrt{x^4+3} + \sqrt{(e^x)^4+3}\cdot\frac{de^x}{dx}$$

$$\therefore \boxed{\frac{dy}{dx} = -\sqrt{x^4+3} + e^x\sqrt{e^{4x}+3}}$$

[8] (a) If $f(x) \geq 8$ on $[-3,3]$, then $\underbrace{\int_{-3}^{3} f(x)\,dx \geq 8[3-(-3)]}_{\text{By properties of definite integrals}} \boxed{= 48}$

(b) If $f(x) \geq 8$ on $[-3,3]$, it tells us nothing about possible values of $f'(x)$ on $[-3,3]$. See example in sketch.



(c) Given $f(-3) = 8$, $0 \leq f'(x) \leq 1$ on $[-3,3]$

Since f is differentiable, by the Mean Value Theorem,

$$0 \leq \frac{f(3)-f(-3)}{3-(-3)} \leq 1 \Rightarrow f(3) - f(-3) \leq 6 \Rightarrow f(3) \leq 6 + f(-3)$$

$\therefore$ The maximum possible value of $\boxed{f(3) = 14}$

(d) Since f is differentiable on $[-3,3]$, and the absolute maximum at $x=1$ would also be a local maximum (since it is not an end point), it follows that $f'(1) = 0$

[8] (e) Linear approximations have the form: $L(x) = f(a) + f'(a)(x-a)$

$$f(1.8) \approx L(1.8) = f(2) + f'(2)(1.8-2) = -0.5 - 0.5(-0.2) = -0.4$$

Answer: $\boxed{f(1.8) \approx -0.4}$

Grading Notes

[1] (a) = (b) = 2 points
For each case, 1/2 pt for minimum axes labels, 1.5 pt. for correct graph

[2] (a) = 2 pt., (b) = 4 points
(a) 1 pt for each correct part of derivative
(b) 1 pt = get $\ln y = x \cdot \ln(\ln x)$; 1 pt = correct derivative of $\ln y$.
1 pt = correctly apply product rule to other side; 1 pt = correct final answer

[3] (a) = (b) = 3 points
(a) 2 pt = get correct Δx and four x_i values; 1 pt = put together correct result
(b) 1.5 pt = get correct Δx and expression for x_i; 0.5 pt = correct bounds on Σ; 1 pt = plug in everything and state correct form of R_n

[4] (a) = 4 points, (b) = 2 points
(a) 0.5 pt = include $\lim_{n\to\infty}$; 0.5 pt = correct bounds on Σ; 1 pt each for following: ① expression inside Σ; ② expression for Δx; ③ correct x_i
(b) 1 pt = Valid example of def int + antiderivative; 1 pt = explain

[5] 1 pt = correct y'; 1.5 pt = correct 3 critical points; 1.5 pt = correct sign analysis for local extremes; 2 pt = correct strategy leading to abs. extremes.

[6] (a) = 2 points, (b) = 4 points
(a) 1.5 pt = correct interpretation; 0.5 pt = correct units
(b) 1 pt = attempt to find indef. int. and to plug in bounds
2 pt = find correct indef. int.; 1 pt = correctly plug in bounds & get answer

[7] 1 pt = correct continuity argument before applying FTC. 2 pt = correctly split integral & flip bounds; 1 pt = correct derivative of "easy" term; 2 pt = correct derivative of other term.

[8] (a) = (b) = (c) = (d) = (e) = 2 points
For each case, 1 pt for correct answer; 1 pt for justification/steps.