Student name:

MATH 180: Calculus A — **Test 2**
Fall 2019 — **October 22, 2019**

Instructions:

- This is a regular "closed-book" test, and is to be taken without the use of notes, books, or other reference materials. Collaboration or group work is not permitted.
- Cell-phone usage in any form is prohibited for the entire duration of the test. This also applies to any restroom breaks taken during the test.
- Answer all questions on separate paper (not on this sheet!).
- Solve all problems using algebra, except if specifically indicated otherwise. Show all solution steps, give reasons, and simplify your answer to receive full credit.
- The time limit for taking this test is 80 minutes from the scheduled start time.
- This test adds up to 53 points. I'll count it as 50, plus 3 bonus points!
- It contains questions numbered (1) through (6).

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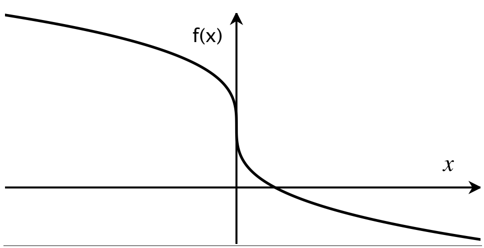
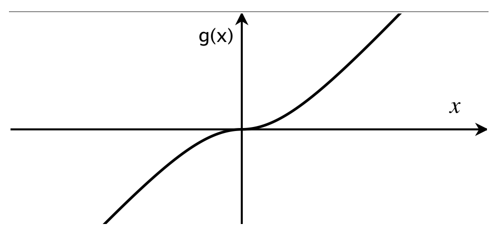
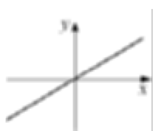
(1) [4 pts.] The graphs of two different functions $f(x)$ and $g(x)$ are shown below:





For each function, identify the graph of its derivative from the following (give reasons):

(A)

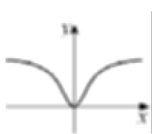


(B)

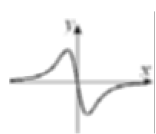
(C)

(D)

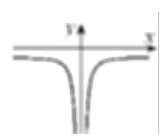
(E)



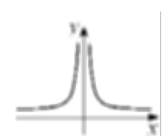
(F)



(G)



(H)



(2) [4 pts.] For each question below, sketch the graph of a function f that has all the indicated properties or, if that is impossible, explain why. As always axes must be completely & clearly labeled for credit.

(a) $\lim_{x\to 2} f(x) = 1$, $f(2) = 2$ and $f'(2) = 0$.

(b) $\lim_{x\to 2} f(x) = 0$, $f(2) = 0$ and $f'(2) =$undefined.

(3) [4 pts.] (a) An aircraft takes-off and ascends in a flight path that is a straight line. Suppose $h(x)$ denotes its altitude (in feet) as a function of distance from the airport (in miles). Give a quantitatively precise explanation of what $h'(120) = 100$ means in this context, and give its units.

(b) Suppose $f(x) = (\sin x)^x$. Use both limit-based definitions of the derivative to set up formulas for finding $f'(x)$. You don't need to actually evaluate the limit and/or find the derivative.

(4) [6 pts.] Find the equation of the line(s) tangent to the graph of $y = \dfrac{x-3}{x+6}$ and parallel to the line $x - 4y + 3 = 0$.

(5) [6 pts.×4] For each of the following, find $\frac{dy}{dx}$ and simplify:

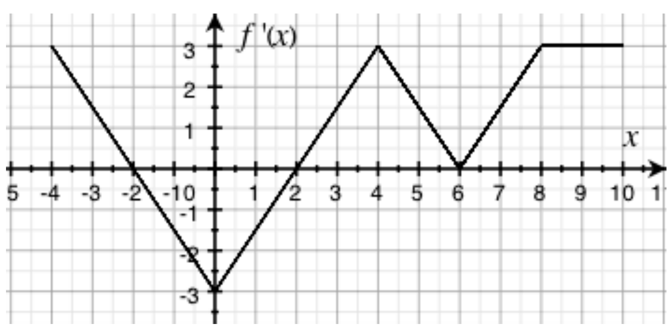
(a) $y = \dfrac{8\sqrt{x} - 4x^{7/2} + 3x\sqrt{x}}{x\sqrt{x}}$

(b) $x = t\sin(t),\ y = e^{\cos^2(t)}$

(c) $e^{y/x} - x^3y = y^3x$

(d) $y = e^{|2x-3|}$

(6) [8 pts.] Shown here is the graph of $f'(x)$, the derivative of some function $f(x)$. Based on this graph, answer the following questions (assume the graph continues to infinity on both ends in the direction shown):



(a) At what x-values does f have local minimum and maximum values? Reason?

(b) On what interval(s) is f concave up, and on what interval(s) is it concave down?

(c) What kind of concavity does f have for $x > 8$? Explain.

(d) Is there enough information here to tell whether $f(4)$ is less than, or greater than, $f(7)$? Explain your answer.

Bonus [3 pts.] The position of an object at time t is given by $s(t) = a\,e^{-bt}\cos(c\,t + d)$, where a, b, c and d are constants. Find the velocity function. (Simplify your answer.)

End of test

Calculus A: Fall 2019: Test 2 Solutions

[1] $f'(x) =$ Graph G. Reason: f is decreasing for all x, so f' must be negative everywhere. Only (G) satisfies this.

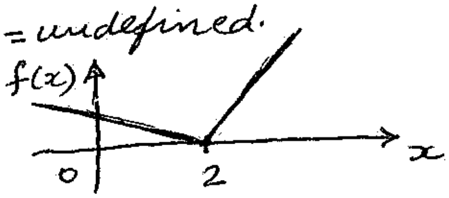
$g'(x) =$ Graph E. Reason: g is increasing for all $x \Rightarrow g'$ is positive, except at $x=0$ where $g'=0$. Only graph (E) has these attributes.

[2] (a) $\lim_{x\to 2} f(x) = 1,\ f(2) = 2,\ f'(2) = 0.$

This is impossible. Reason: Since $\lim_{x\to 2} f(x) \neq f(2)$, the function is discontinuous at $x=2$. That means f' is undefined at $x=2$.

(b) $\lim_{x\to 2} f(x) = 0,\ f(2) = 0,\ f'(2) =$ undefined.

one example of such a function is shown in the graph.



[3] $h'(120) = 100$ means: When the aircraft is 120 miles from the airport,

(a) its altitude is changing at an instantaneous rate of 100 feet/mile with respect to distance from the airport.

(b) Given $f(x) = (\sin x)^x$. The definition of the derivative at $x=a$ is

$$f'(a) = \lim_{h\to 0} \frac{[\sin(a+h)]^{(a+h)} - [\sin a]^a}{h}$$

OR $$f'(a) = \lim_{x\to a} \frac{(\sin x)^x - (\sin a)^a}{x-a}$$

[4] Want eqn. of tangent to $y = \frac{x-3}{x+6}$ and parallel to $x - 4y + 3 = 0$

Rewrite line in standard form: $y = \frac{x+3}{4} = \frac{1}{4}x + \frac{3}{4}$

Since we want tangent parallel to this line, slope of tangent $\boxed{= \frac{1}{4}}$

Next, let's find the x-location(s) where $y' = \frac{1}{4}$ for $y = \frac{x-3}{x+6}$

$$y' = \frac{(x+6)(x-3)' - (x-3)(x+6)'}{(x+6)^2} = \frac{(x+6) - (x-3)}{(x+6)^2} = \boxed{\frac{9}{(x+6)^2}}$$

Set $y' = \frac{1}{4}$ and solve for x: $\frac{9}{(x+6)^2} = \frac{1}{4} \Rightarrow 36 = (x+6)^2$

$\Rightarrow \pm 6 = x + 6 \Rightarrow \boxed{x = 0,\ x = -12}$

$\therefore$ There are 2 x-values where $y' = \frac{1}{4}$

The points of tangency are: $y_1 = \frac{0-3}{0+6} = -\frac{1}{2}$, $y_2 = \frac{-12-3}{-12+6} = \frac{-15}{-6} = \frac{5}{2}$

point 1 is $(0, -1/2)$

point 2 is $(-12, 5/2)$

Equations of tangent lines:

Line 1: $y = \frac{1}{4}x + b \Rightarrow -\frac{1}{2} = \frac{1}{4}(0) + b \Rightarrow b = -1/2$

$\therefore \boxed{y = \frac{1}{4}x - \frac{1}{2}}$ ← Line 1

Line 2: $y = \frac{1}{4}x + b \Rightarrow \frac{5}{2} = -3 + b \Rightarrow b = \frac{11}{2}$ $\therefore \boxed{y = \frac{1}{4}x + \frac{11}{2}}$ ← Line 2

[5] (a) $y = \dfrac{8\sqrt{x} - 4x^{7/2} + 3x\sqrt{x}}{x\sqrt{x}}$. Simplify & get $y = 8x^{-1} - 4x^2 + 3$.

$\therefore \boxed{y' = -\dfrac{8}{x^2} - 8x} = \dfrac{-8(1+x^3)}{x^2}$

(b) $x = t \cdot \sin(t)$, $y = e^{\cos^2(t)}$

$\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$. Here we have $\dfrac{dx}{dt} = t \cdot \dfrac{d}{dt}\sin(t) + \sin(t) \cdot \dfrac{dt}{dt}$

$\Rightarrow \boxed{\dfrac{dx}{dt} = t \cdot \cos(t) + \sin(t)}$

$\dfrac{dy}{dt} = e^{\cos^2(t)} \cdot \underbrace{\dfrac{d}{dt}\cos^2(t)}_{= 2 \cdot \cos(t) \cdot \frac{d}{dt}\cos(t) = 2\cos(t) \cdot (-\sin t)}$

$\Rightarrow \boxed{\dfrac{dy}{dt} = -2\sin(t) \cdot \cos(t) \cdot e^{\cos^2(t)}}$

$\therefore \boxed{\dfrac{dy}{dx} = -\dfrac{2\sin(t) \cdot \cos(t) \cdot e^{\cos^2(t)}}{\sin(t) + t \cdot \cos(t)}}$

(c) $e^{y/x} - x^3 y = y^3 x$

All primes denote derivatives with respect to x in the work that follows. I'll differentiate both sides of the eqn. with respect to x:

$e^{y/x}\left[\dfrac{x \cdot y' - y}{x^2}\right] - \left[3x^2 y + x^3 y'\right] = 3y^2 \cdot y' \cdot x + y^3$

To solve for y', I'll multiply through by x^2, then solve:

$e^{y/x}[x \cdot y' - y] - 3x^4 y - x^5 y' = 3y^2 \cdot x^3 \cdot y' + x^2 y^3$

$\Rightarrow y'[x \cdot e^{y/x} - x^5 - 3y^2 x^3] = x^2 y^3 + y e^{y/x} + 3x^4 y$

$\therefore \boxed{\dfrac{dy}{dx} = y' = \dfrac{x^2 y^3 + 3x^4 y + y \cdot e^{y/x}}{-x^5 - 3x^3 y^2 + x \cdot e^{y/x}}}$

(d) $y = e^{|2x-3|}$. Rewrite as piecewise fn: $y = \begin{cases} e^{2x-3}, & \text{if } x \geq \frac{3}{2} \\ e^{3-2x}, & \text{if } x < \frac{3}{2} \end{cases}$

Differentiate each piece on its domain

$y' = \begin{cases} 2e^{2x-3}, & \text{if } x > \frac{3}{2} \\ -2e^{3-2x}, & \text{if } x < \frac{3}{2} \end{cases}$

Must look at left/right limits when $x = 3/2$

$\left.\begin{array}{l} \text{From the right: } y' = 2e^{2(3/2)-3} = 2 \cdot e^0 = 2 \\ \text{From the left: } y' = -2e^{3-2(3/2)} = -2e^0 = -2 \end{array}\right\}$ Not equal $\Rightarrow y' = DNE$ at $x = 3/2$

Answer: If $x > \frac{3}{2}$, $y' = 2e^{2x-3}$

If $x < \frac{3}{2}$, $y' = -2e^{3-2x}$, if $x = \frac{3}{2}$, y' DNE

[6] (a) f has local maximum at $x=-2$. Reason: f' goes from + to −
It has local minimum at $x=2$. Reason: f' goes from − to +

(b) f is concave up on: $(0,4)$ and $(6,8)$
concave down on: $(-\infty,0)$ and $(4,6)$
Reason: concave up when f' is increasing; down when f' is decreasing

(c) For $x>8$, f' is constant, which means the graph of f will be a straight line with slope $=3$ (because $f'=3$). Thus, it is neither concave up, nor down for $x>8$.

(d) Yes. $f(4)$ is less than $f(7)$. Reason: Between $x=4$ and 7, $f'>0$ (except at $x=6$, where $f'=0$). That means f is increasing on that interval. So $f(7)$ will have to be larger than $f(4)$.

Bonus: Position function is $s(t)=ae^{-bt}\cos(ct+d)$, with a, b, c, d constant

$$\text{Velocity} = s'(t) = (a\cdot e^{-bt})'\cdot\cos(ct+d) + a\cdot e^{-bt}[\cos(ct+d)]'$$

$$= -a\cdot b\cdot e^{-bt}\cdot\cos(ct+d) + a\cdot e^{-bt}[-\sin(ct+d)\cdot c]$$

$$\boxed{\text{Velocity} = -a\cdot e^{-bt}[b\cdot\cos(ct+d) + c\cdot\sin(ct+d)]}$$

Grading notes:

[1] 2 points each for correct f' and g'. (1.5 pt for answer, 0.5 pt for reason)
[2] (a) = 2 points, (b) = 2 points → For each, 50/50 split between answer & reason.
[3] (a) = 2 points, (b) = 2 points
For (a): Answer must convey ① rate of change, ② of altitude, ③ with respect to distance, ④ at distance of 120; ⑤ with correct units.
For (b): 1 pt. each for the 2 definitions — no partial credit
[4] 1.5 pt = compute correct derivative y'; 1.5 pt = set y' to $\frac{1}{4}$ and get $x=0$, $x=-12$;
1 pt = find y-values matching the x's; 1+1 pt = correct tangent line equations
[5] (a) 4 pt = correctly simplify to powers of x; 2 pt = correct derivative of result.
[If student uses Q.R. instead, −2 pt if result not completely simplified]
(b) 1 pt = Know that $dy/dx = (dy/dt)/(dx/dt)$
2+2 pt = correct derivatives dy/dt and dx/dt
1 pt = correctly put together and get dy/dx
(c) 2 pt = correctly differentiate $e^{y/x}$; 1.5+1.5 pt = correctly diff. other 2 terms
1 pt = correctly solve and get dy/dx.
(d) 1 pt = rewrite y piecewise, with correct domains;
1.5+1.5 pt = Correctly differentiate each piece; 1 pt = show why y' DNE at $x=\frac{3}{2}$;
1 pt = put everything together & state correct final result for y'.
[6] (a) = (b) = (c) = (d) = 2 points each.
[Bonus] 0.5 pt = Know that $V=s'$; 1.5 pt = correctly differentiate whole expression,
1 pt = simplify the result by factoring out $a\cdot e^{-bt}$