Student name:

MATH 180: Calculus A | **Test 1**
Fall 2019 | **September 17, 2019**

Instructions:

- This is a regular "closed-book" test, and is to be taken without the use of notes, books, or other reference materials. Collaboration or group work is not permitted.
- Cell-phone usage in any form is prohibited for the entire duration of the test. This also applies to any restroom breaks taken during the test.
- Answer all questions on separate paper (not on this sheet!).
- Solve all problems using algebra, except if specifically indicated otherwise. Show all solution steps, give reasons, and simplify your answer to receive full credit.
- The time limit for taking this test is 80 minutes from the scheduled start time. Please turn in your test promptly when time is called to avoid late penalties.
- This test adds up to 50 points. It contains questions numbered 1 through 7.

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(1) [4 pts.] Find the inverse of the function $v(t) = \sqrt{4 - e^{-2t}}$.

(2) [4 pts.] Construct an example of a function that has vertical asymptotes at $x = -2$, $x = 0$ and $x = 1$, and a horizontal asymptote at $y = 2$. Give your function in algebraic form and give reasons or show steps.

(3) [6 pts.] Given the function

$$f(x) = \begin{cases} x+1, & \text{if } x \le -1 \\ x^2 - 1, & \text{if } |x| < 1 \\ \ln(x), & \text{if } x \ge 1 \end{cases}$$

Determine where the function is continuous and where it is discontinuous. Justify all claims using the mathematical definition of continuity or by using relevant theorems.

(4) [6 pts. each $\times$ 3] Evaluate the following limits using algebra and showing all steps. If a limit fails to exist, be sure to determine whether it is ∞, $-\infty$, or some other form of DNE.

(a) $\displaystyle\lim_{x \to 2} \frac{\frac{2}{x^2} - \frac{1}{2}}{x - 2}$

(b) $\displaystyle\lim_{x \to -2} \frac{x^2 + 4x + 3}{x^2 + 5x + 6}$

(c) $\displaystyle\lim_{x \to 4} \frac{|x^2 - 4x|}{4 - x}$

(5) [6 pts.] Solve for x: $\quad 2\log(x+1) = \log(2x-1) + \log(x-1)$

(6) [6 pts.] Sketch the graph of a function f with the following properties

$$\lim_{x\to -\infty} f(x) = -2, \lim_{x\to -3} f(x) = \infty, \lim_{x\to 1^-} f(x) = -1, \lim_{x\to 1^+} f(x) = 2, f(1) = 0,$$

Graph must include detailed labels, and indicate open/closed intervals as needed.

(7) [6 pts.] (a) Give a complete and mathematically precise statement of the Intermediate Value Theorem. Include a sketch to illustrate the theorem's assertion.

(b) Show that the equation $(2-x)e^x = x-1$ has a root (i.e., a solution) by applying the Intermediate Value Theorem.

End of test

Calculus A: Fall 2019: Test 1 Solutions

[1] To find the inverse of $V(t) = \sqrt{4 - e^{-2t}}$, we can just solve the equation for t.

$$V = \sqrt{4 - e^{-2t}} \Rightarrow V^2 = 4 - e^{-2t} \Rightarrow e^{-2t} = 4 - V^2$$

Taking ln on both sides, we get: $-2t = \ln(4 - V^2)$.

$$\therefore t = -\frac{\ln(4-V^2)}{2}$$

Answer: $\boxed{t = -\frac{\ln(4-V^2)}{2}}$

May switch the variables, or leave them as is.

[2] The easiest way to do this is via a suitably chosen rational function.

Vertical asymptotes at $x = -2, 0, 1 \Rightarrow$ denominator $= 0$ at those points

One way to get 0 denominator at those points: $\frac{1}{(x+2)\,x\,(x-1)}$

Since the denominator is cubic (i.e, $x^3 + \cdots$), and we want a horizontal asymptote at $y = 2$, the numerator must have $2x^3$ as its leading term. So, here is one function that works:

$$\boxed{f(x) = \frac{2x^3 + 1}{(x+2)\,x\,(x-1)}}$$

Another option: $g(x) = 2 + \frac{1}{(x+2)x(x-1)}$

[3]

$$f(x) = \begin{cases} x+1, & \text{if } x \le -1 \\ x^2 - 1, & \text{if } |x| < 1 \\ \ln(x), & \text{if } x \ge 1 \end{cases}$$

I will consider 5 parts of the function's potential domain.

Part 1: $x < -1$. On this piece, $f(x) = x+1$ which is continuous by theorem that says all polynomials are continuous.

Part 2: $-1 < x < 1$: Here $f(x) = x^2 - 1$, which is continuous by theorems.

Part 3: $x > 1$: On this piece $f(x) = \ln(x)$, which is continuous on its entire domain by theorems.

Part 4: $x = -1$: Here I will check whether the definition is satisfied.

$f(-1) = (-1) + 1 = 0$. $\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} (x+1) = 0$, $\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} (x^2 - 1) = 0$ $\Big\}\ \lim_{x \to -1} f(x) = 0$

It is continuous at $x = -1$, because $f(-1) = \lim_{x \to -1} f(x)$

Part 5: $x = 1$: $f(1) = \ln(1) = 0$. $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x^2 - 1) = 0$

$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (\ln x) = \ln(1) = 0$

$\therefore f(1) = \lim_{x \to 1} f(x) = 0$. So it is continuous at $x = 1$

Ans: The function in Q.3 is continuous for all x.

[4] (a) $\lim_{x\to 2} \frac{\frac{2}{x^2}-\frac{1}{2}}{x-2}$. Plug in gives $\frac{\frac{2}{4}-\frac{1}{2}}{2-2} \sim \frac{0}{0} \Rightarrow$ doesn't work

Try to cancel $(x-2)$ factor via algebra:

$$\frac{4-x^2}{2x^2} \div (x-2) = \frac{(2-x)(2+x)}{2x^2(x-2)} = -\frac{(2+x)}{2x^2} \Rightarrow \frac{-2+2}{2\cdot 2^2} = -\frac{4}{8}$$

$$\lim_{x\to 2}\left(\text{given stuff}\right) = \lim_{x\to 2} -\frac{(2+x)}{2x^2} = \boxed{-\frac{1}{2}} \text{ Answer}$$

(b) $\lim_{x\to -2} \frac{x^2+4x+3}{x^2+5x+6}$. Plugging in $x=-2$ we get: $\frac{4-8+3}{4-10+6} = -\frac{1}{0}$

Try to cancel $(x+2)$ by doing algebra.

$$\frac{x^2+4x+3}{x^2+5x+6} = \frac{\cancel{(x+3)}(x+1)}{(x+2)\cancel{(x+3)}} = \frac{x+1}{x+2} \Rightarrow \text{cannot cancel out } (x+2)$$

Thus the limit as $x\to -2$ will be some sort of DNE.

Left-limit: $\lim_{x\to -2^-} \frac{x+1}{x+2} \sim \frac{-1}{-0} = +\infty$

Right limit: $\lim_{x\to -2^+} \frac{x+1}{x+2} \sim \frac{-1}{+0} = -\infty$

$\boxed{\text{Answer: Left limit } = +\infty \text{, Right limit } = -\infty}$

(c) $\lim_{x\to 4} \frac{|x^2-4x|}{4-x}$. The numerator is $|x^2-4x| = |x(x-4)|$

$$= \begin{cases} -x(x-4) & \text{if } 0<x<4 \\ x(x-4), & \text{otherwise} \end{cases}$$

Left-limit: $\lim_{x\to 4^-} \frac{|x^2-4x|}{4-x}$

$$= \lim_{x\to 4^-} \frac{-x(x-4)}{4-x} = \lim_{x\to 4^-} x = 4$$

Right limit $= \lim_{x\to 4^+} \frac{|x^2-4x|}{4-x} = \lim_{x\to 4^+} \frac{x(x-4)}{4-x} = \lim_{x\to 4^+} (-x) = -4$

The left limit $= 4$ and right limit $= -4$. Thus the limit DNE.

[5] $2\log(x+1) = \log(2x-1) + \log(x-1)$

$\Rightarrow \log(x+1)^2 = \log[(2x-1)(x-1)]$

$\Rightarrow (x+1)^2 = (2x-1)(x-1)$

$\Rightarrow x^2+2x+1 = 2x^2-3x+1 \Rightarrow 0 = x^2-5x = x(x-5) \therefore x=0,\ x=5$

Plugging in $x=0$ shows that it doesn't work.

So, the only solution is $\boxed{x=5}$

[6]

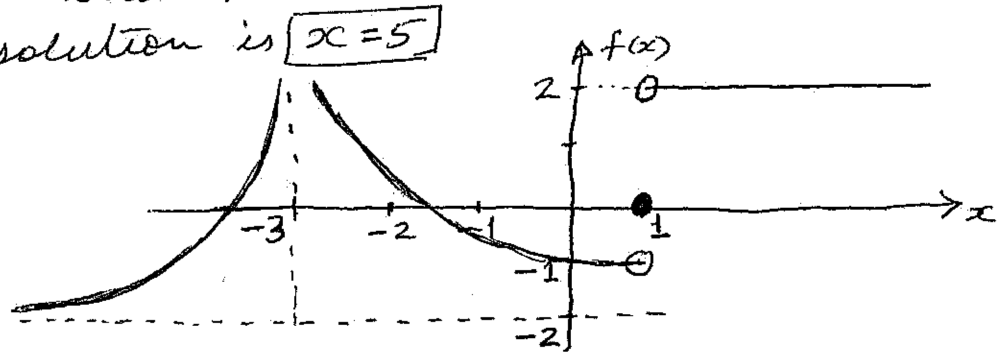
$\lim_{x\to -\infty} f(x) = -2$

$\lim_{x\to -3} f(x) = \infty$

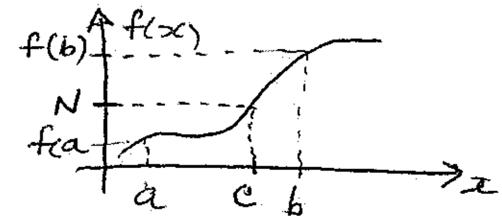
$\lim_{x\to 1^-} f(x) = -1$

$\lim_{x\to 1^+} f(x) = 2$

$f(1) = 0$



[7] I.V.T. Let f be a continuous function on the interval $[a, b]$, with $f(a) \neq f(b)$. Then, for any number N between $f(a)$ and $f(b)$, there exists a number c between a and b where $N = f(c)$.



(b) $(2-x)e^{x} = x-1$

Let $f(x) = (2-x)e^{x} - x + 1$

All functions involved in composing $f(x)$ are continuous for all x values, by continuity theorems. Therefore, we can apply the I.V.T. to $f(x)$ on any interval.

$f(0) = 2e^{0} - 0 + 1 = 3, \quad f(2) = (2-2)e^{2} - 2 + 1 = -1$

Applying the I.V.T. on $[0, 2]$, since $f(0) > 0$ and $f(2) < 0$, it follows that $f(x) = 0$ for some x between 0 and 2. Therefore, the given equation has a root.

Grading Notes

[1] 1 pt = square both sides; 1 pt = attempt to apply ln after rearranging
2 pt = do it correctly and simplify

[2] 2 pt = an expression that contains the right denominator for vertical A.
1 pt = get correct numerator; 1 pt = show some reasoning or justification

[3] 3 pt = justify & state continuous on the 3 open intervals.
3 pt = correctly determine continuity at boundary points (1.5 pt. each)

[4] (a) 3 pt = common denom. the numerator & split into 2 factors
2 pt = simplify to $-(2+x)/(2x^2)$; 1 pt = plug in $x=2$ and get answer

(b) 3 pt = correct LL, with justification or steps
3 pt = correct RL, " " "
[If student factors & cancels $(x+3)$ only, and doesn't get other things right, they get 1 point]

(c) 1 pt = attempt to find left/right limits,
2 pt = find correct left limit; 2 pt = find correct right limit
1 pt = conclude that limit DNE.

[5] 2 pt = combine the logs in any correct & useful way
1 pt = drop logs and get valid algebraic eqn.
2 pt = correctly solve algebraic eqn.
1 pt = check solutions & figure out which one works

[6] 1 pt each = satisfy each of the 5 requirements via a valid function
1 pt = axes labels, open/closed circles, etc.

[7] (a) = (b) = 3 points each
For (a): 1 pt = correct hypothesis; 1.5 pt = correct conclusion; 0.5 pt = sketch
For (b): 1 pt = turn given eqn. into a function; 0.5 pt = check for continuity
1.5 pt = find 2 points where f has opposite signs & draw conclusion.