Quiz 7 - 11/01/2019

(I) Find dy/dx for: $x^y = y^x$

(II) Linearize the function $f(x) = x^{1/3}$ around the point $x = 8$.

Solution

(I) Take "ln" on both sides in order to move variables out of the exponents:

$$\ln(x^y) = \ln(y^x) \quad \Rightarrow \quad y\ln(x) = x\ln(y)$$

Next, differentiating both sides with respect to x, we have

$$\frac{dy}{dx}\ \ln(x) + y\ \frac{d(\ln x)}{dx} = \frac{dx}{dx}\ \ln(y) + x\ \frac{d(\ln y)}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx}\ \ln(x) + \frac{y}{x} = \ln(y) + x\ \left(\frac{1}{y}\right)\ \frac{dy}{dx}$$

Solve for dy/dx:

$$\frac{dy}{dx}\ \left(\ln(x) - \frac{x}{y}\right) = \ln(y) - \frac{y}{x}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\ln(y) - \frac{y}{x}}{\ln(x) - \frac{x}{y}}$$

$$\therefore \quad \boxed{\frac{dy}{dx} = \frac{xy\ln(y) - y^2}{xy\ln(x) - x^2}}$$

(II) Given $f(x) = x^{1/3} \ \Rightarrow \ f'(x) = (1/3)x^{-2/3}$

Linear approximations have the form: $L(x) = f(a) + f'(a)(x - a)$

In this problem $a = 8$. Thus: $L(x) = (8)^{1/3} + \frac{1}{3}(8)^{-2/3}(x - 8)$

Answer: $\boxed{L(x) = 2 + \frac{1}{12}(x - 8) \quad \text{OR} \quad L(x) = \frac{x}{12} + \frac{4}{3}}$

Grading: Total points possible = 6.

4 pt for (I): 1pt = correctly apply log prop and rewrite as $y\ln(x) = x\ln(y)$
2pt = correct derivatives of the resulting two sides.
1pt = simplify (get result without fractions inside fractions).

2 pt for (II): 1pt = find correct $f'(x)$ and $f'(8)$.
1pt = plug everything into formula and get right answer.