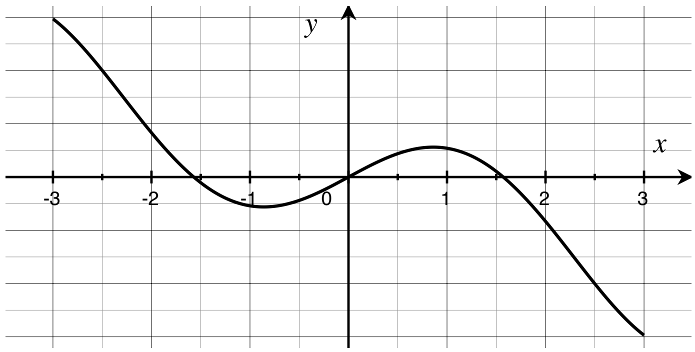
Quiz 4 - 9/27/2019

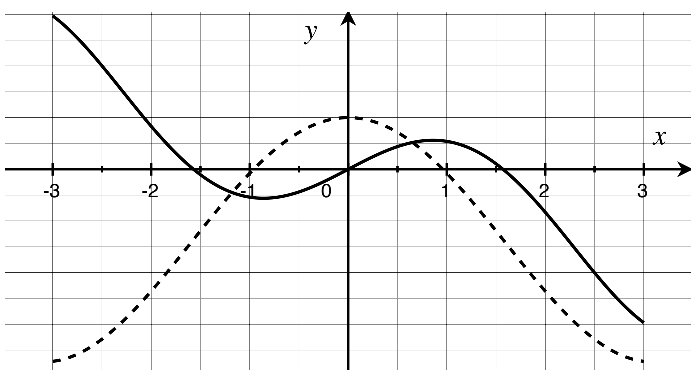
(I) The graph of some function $y = f(x)$ is shown. Sketch the graph of $f'(x)$. You may do this directly on the graph below. Include a short discussion to justify why your graph of $f'(x)$ is the right solution.



(II) Use the definition of derivative to find an expression for the derivative of $f(x) = 1/\sqrt{x}$ at $x = a$. (NOTE: The goal of this question is to assess your understanding of the definition and ability to apply it. Thus, if you know other methods of finding the correct answer, it would be pointless using them here!)

Solution

(I) The dotted curve shown below is the graph of $f'(x)$



Resaoning: f' is 0 wherever the tangent line is horizontal on $f(x)$. There are 2 places where this occurs: $x \approx -1$ and $x \approx 1$. Our f' graph intersects the x- axis at those points. Also note that f is decreasing wherever f' is negative, and f is increasing wherever f' is positive.

(II) By definition, the derivative of $f(x)$ at any $x = a$ is: $f'(a) = \lim_{x \to a} \dfrac{f(x) - f(a)}{x - a}$.

Plugin the given $f(x)$ and get $f'(a) = \lim_{x \to a} \dfrac{\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{a}}}{x - a}$.

Try to find $(x - a)$ cancellation.

Simplify the numerator: $\dfrac{1}{\sqrt{x}} - \dfrac{1}{\sqrt{a}} = \dfrac{\sqrt{a}}{\sqrt{x} \cdot \sqrt{a}} - \dfrac{\sqrt{x}}{\sqrt{x} \cdot \sqrt{a}} = \dfrac{\sqrt{a} - \sqrt{x}}{\sqrt{x} \cdot \sqrt{a}}$.

Plug into previous step: $f'(a) = \lim_{x \to a} \left[\dfrac{\sqrt{a} - \sqrt{x}}{\sqrt{x} \cdot \sqrt{a}} \div (x - a) \right] = \lim_{x \to a} \left[\dfrac{\sqrt{a} - \sqrt{x}}{\sqrt{x} \cdot \sqrt{a}} \cdot \dfrac{1}{(x - a)} \right]$.

Rationalize: multiply and divide by $\sqrt{a} + \sqrt{x}$

$$= \lim_{x \to a} \left[\frac{\sqrt{a} - \sqrt{x}}{\sqrt{x} \cdot \sqrt{a} \cdot (x-a)} \right] \cdot \left[\frac{\sqrt{a} + \sqrt{x}}{\sqrt{a} + \sqrt{x}} \right] = \lim_{x \to a} \left[\frac{(a-x)}{\sqrt{x} \cdot \sqrt{a} \cdot (x-a) \cdot (\sqrt{a} + \sqrt{x})} \right]$$

$$= \lim_{x \to a} \left[\frac{-1}{\sqrt{x} \cdot \sqrt{a} \cdot (\sqrt{a} + \sqrt{x})} \right]$$

Now plugin $x = a$: $f'(a) = \left[\frac{-1}{\sqrt{a} \cdot \sqrt{a} \cdot (\sqrt{a} + \sqrt{a})} \right] = -\frac{1}{a\,(2\sqrt{a})}$

Answer: $\boxed{f'(x) = -\frac{1}{2x^{3/2}}}$

Grading: Total points possible = 6.

2.5 pt for (I): 1 pt = graph of derivative intersects x-axis at the right places.
1 pt = correct sign throughout.
0.5 pt = explanation..

3.5 pt for (II): 0.5 pt = know/show a correct formula for $f'(a)$.
0.5 pt = correctly plug the given $f(x)$ into that formula.
2 pt = correct algebra, up to $\frac{-1}{\sqrt{x} \cdot \sqrt{a} \cdot (\sqrt{a} + \sqrt{x})}$.
0.5 pt = plug in $x = a$ and get correct answer.