Quiz 2 - 9/03/2019

(I) Sketch the graph of a function f that satisfies all the following requirements:

$$\lim_{x \to 0} f(x) = 2, \quad f(0) = 2$$

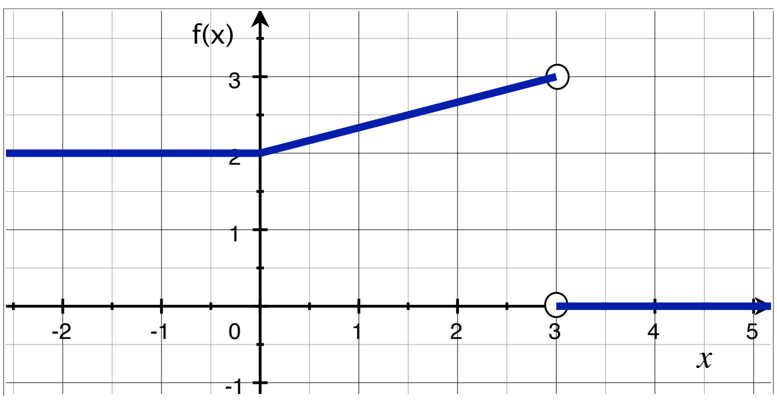
$$\lim_{x \to 3^-} f(x) = 3, \quad \lim_{x \to 3^+} f(x) = 0, \quad f(3) = \text{undefined}$$

As always, graph must show all labels and symbols needed to read it correctly.

(II) Find a formula for the inverse of the function $f(x) = \dfrac{e^x}{1 - 2e^x}$. Show steps.

Solution

(I) At $x = 0$ there is no hole or break in the graph, because we want the limit to be the same as the function value. At $x = 3$ the graph must break, to accommodate unequal limits on the left and right. There are many possible correct solutions to this problem. One example of a function that satisfies all the requirements is shown below



(II) To find the inverse, we switch the roles of x, y and solve for y, (in terms of x).

Switch x, y and get: $x = \dfrac{e^y}{1 - 2e^y}$

Multiply both sides by $1 - 2e^y$: $x(1 - 2e^y) = e^y \Rightarrow x - 2xe^y = e^y$

Group all y-terms on one side: $x = e^y + 2xe^y$

Factor out e^y and divide through by factor: $x = e^y(1 + 2x) \Rightarrow \dfrac{x}{1 + 2x} = e^y$

Take ln on both sides: $\ln\left[\dfrac{x}{1 + 2x}\right] = y$

Answer: $\boxed{y = f^{-1}(x) = \ln\left[\dfrac{x}{1 + 2x}\right]}$

Verify answer (not a required part of the solution):

$$f(f^{-1}(x)) = \frac{e^{f^{-1}(x)}}{1 - 2e^{f^{-1}(x)}} = \frac{e^{\ln[x/(1+2x)]}}{1 - 2e^{\ln[x/(1+2x)]}} = \frac{x/(1 + 2x)}{1 - 2x/(1 + 2x)} = \frac{x}{(1 + 2x) - 2x} = x$$

Grading: Total points possible = 6.

3 pt for (I): 1pt = correct graph at/around $x = 0$
plus 0.5pt for each of the following 4 features:
(a) correct left-limit at $x = 3$, including open circle
(b) correct right-limit at $x = 3$, including open circle
(c) correctly leave $f(3)$ undefined
(d) graph shows all needed axes labels

3 pt for (II): 0.5pt = Attempt to flip x, y and solve for y.
1.5pt = correct algebraic steps till getting e^y.
0.5pt = take ln and get y.
0.5pt = correctly express final result in the form $y = \cdots$, or $f^{-1}(x) = \cdots$.