Student name:

MATH 180: Calculus A | **Final exam**
Fall 2019 | **December 9, 2019**

Instructions:

- This is a regular "closed-book" test, and is to be taken without the use of notes, books, or other reference materials. Collaboration or group work is not permitted.
- Cell-phone usage in any form is prohibited for the entire duration of the test. This also applies to any restroom breaks taken during the test.
- Answer all questions on separate paper (not on this sheet!).
- Solve all problems using algebra, except if specifically indicated otherwise. Show all solution steps, give reasons, and simplify your answer to receive full credit.
- The time limit for taking this test is 2.5 hours from the scheduled start time.
- This test contains questions numbered 1-7. It adds up to 52 points – I'll count it as 50, plus 2 points bonus!
- Good luck on the test!

---

(1) [4 pts.] Give a complete and mathematically precise statement of the Fundamental Theorem of Calculus. Explain why the theorem is considered significant, or "fundamental."

(2) [5 pts.×2] Find dy/dx for each of the following and simplify:

(a) $y = \dfrac{x^2 - 4x\sqrt{x} + \sin x}{\sqrt{x}}$ (b) $y \ln(x) = \cos(y^2)$

<u>Hint</u>: Don't use quotient rule for (a) – unless you're cool with doing loads of algebra!

(3) [5 pts] Find all the local and absolute extreme values of $f(x) = |x|(4 - x^2)$ on the interval $[-2, 3]$. Credit for correct calculus and algebra steps only – not for answers!

(4) [5 pts] Find the most general f that satisfies: $f'' = 3e^x + 4\cos x - \dfrac{1}{2x^2}$

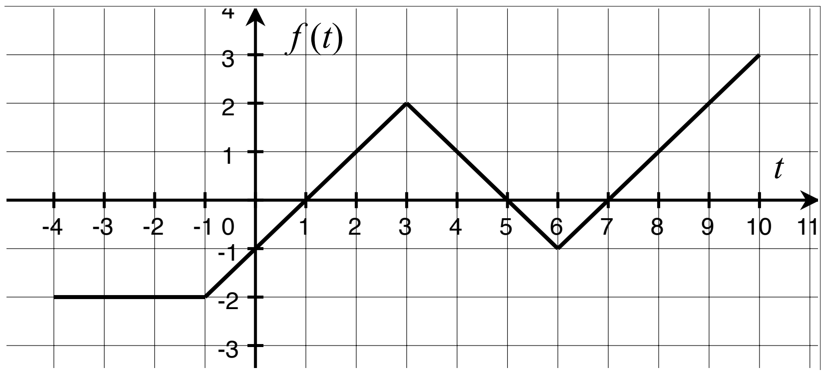
(5) [5 pts.×2] Evaluate the following limits:

(a) $\displaystyle \lim_{x \to -2} \left(\frac{x^2}{x^2 - 4} + \frac{1}{x + 2} \right)$

(b) $\displaystyle \lim_{x \to 0} \frac{\cos^2 x - 1}{x^2}$

Show solution steps and reasons.

(6) [8 pts.] Let $g(x) = \int_1^x f(t)dt$, where the graph of f is shown below.



(a) What is the sign of $g(-1)$, $g(3)$ and $g(6)$ [positive or negative]? Explain your reasoning.

(b) Find the value of $g(5)$ and $g'(5)$. Give reasons.

(c) On $-4 < x < 10$ find the interval(s) on which g is increasing, and those on which it is decreasing.

(d) On $-4 \leq x \leq 10$ find the absolute minimum and maximum values of g. Show steps/ reasoning.

(7) [10 pts.] Give brief answers to each of the following as instructed:

(a) Sketch the graph of a function with all the following properties or, if that is impossible, explain why: continuous everywhere; has two critical points; differentiable at one of those points, but not at the other; one local minimum, no local maximum.

(b) State the limit-based definition of the derivative of a function, and show how to plug in the function $f(x) = g(x) \cdot h(x)$ into your definition.

(c) Suppose $s'(x)$ is the slope of a trail (in feet per mile) as a function of distance from the start of the trail (in miles). What does $\int_2^5 s'(x)dx = 300$ mean in this application? What are its units?

(d) Some function f has the property that $f'(x) > 0$ and $f''(x) < 0$ for all x. Suppose $f(4) = 3$. How many solutions does $f(x) = 0$ have? Explain your answer.

(e) Evaluate the following two expressions (give reasons):

(i) $\int_0^{\pi/2} \frac{d}{dx}(e^{\sin^2 x})\,dx$ (ii) $\frac{d}{dx}\left(\int_0^{\pi/2} e^{\sin^2 x}\,dx\right)$

End of test

Calculus A: Fall 2019: Final Exam Solutions

[1] The Fundamental Theorem of Calculus says: Let f be a continuous function on the interval $[a, b]$. Then, the following two conclusions hold:

(i) $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$, for any x in $[a, b]$

(ii) $\int_a^b f(x)\,dx = F(b) - F(a)$, where F is any antiderivative of f. That is, $F' = f$ on $[a, b]$.

The theorem is considered significant because it links together the 2 major branches of calculus, and suggests an inverse relationship between integration and differentiation. For instance, (i) says that the derivative of the integral of a function gives back the same function.

[2] (a) $y = \dfrac{x^2 - 4x\sqrt{x} + \sin x}{\sqrt{x}} = \dfrac{x^2}{\sqrt{x}} - \dfrac{4x\sqrt{x}}{\sqrt{x}} + \dfrac{\sin x}{\sqrt{x}}$

$$= x^{3/2} - 4x + x^{-1/2}\cdot \sin x$$

$$\therefore \frac{dy}{dx} = \frac{3}{2}x^{1/2} - 4 + \left[x^{-1/2}\cdot \cos x - \frac{1}{2}x^{-3/2}\cdot \sin x\right]$$

$$= \frac{3\sqrt{x}}{2} - 4 + \frac{\cos x}{\sqrt{x}} - \frac{\sin x}{2x\sqrt{x}}$$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{3x^2 - 8x\sqrt{x} + 2x\cdot\cos x - \sin x}{2x\sqrt{x}}}$$

(b) $y\cdot \ln(x) = \cos(y^2)$

Differentiating implicitly with respect to x, we have

$$y\cdot\left(\frac{1}{x}\right) + \frac{dy}{dx}\ln(x) = -\sin(y^2)\cdot 2y\cdot\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx}\left[\ln(x) + 2y\cdot\sin(y^2)\right] = -\frac{y}{x}$$

$$\therefore \boxed{\frac{dy}{dx} = -\frac{y}{x\left[\ln(x) + 2y\cdot\sin(y^2)\right]}}$$

[3] The given function is $f(x) = |x|(4 - x^2) = \begin{cases} x(4-x^2), & \text{if } x \geq 0 \\ -x(4-x^2), & \text{if } x < 0 \end{cases}$

Find the critical points first:

$$f(x) = \begin{cases} 4x - x^3, & \text{if } x \geq 0 \\ -4x + x^3, & \text{if } x < 0 \end{cases} \Rightarrow f'(x) = \begin{cases} 4 - 3x^2, & \text{if } x > 0 \\ -4 + 3x^2, & \text{if } x < 0 \end{cases}$$

$$f' = 0 \Rightarrow 4 - 3x^2 = 0 \Rightarrow x^2 = \frac{4}{3} \Rightarrow x = \pm\frac{2}{\sqrt{3}}$$

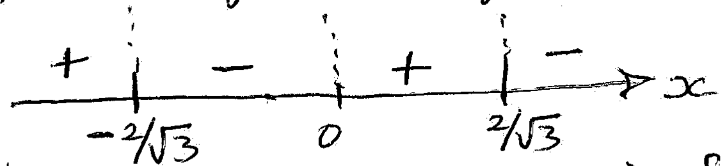
At $x = 0$: f' from the left $= -4$; f' from the right $= 4$

$\therefore f'(0) = DNE$

It follows that there are 3 critical points: $x = \pm \frac{2}{\sqrt{3}}$, $x = 0$

All critical points lie inside the given interval $[-2, 3]$

* For local extremes, make sign chart of f'



* There are two points of local maximum: $f(-2/\sqrt{3}) = f(2/\sqrt{3}) = 3.0972$

There is one local minimum: $f(0) = 0$

* For absolute extremes, make table of function values at c.P.s and end-points:

x	-2	$-2/\sqrt{3}$	0	$2/\sqrt{3}$	3
$f(x)$	0	3.0792	0	3.0792	-15

Answers: Local maximum at: $f(-2/\sqrt{3}) = 3.0792$, $f(2/\sqrt{3}) =$ same

Local minimum at: $f(0) = 0$

Abs. maximum at: $f(-2/\sqrt{3}) = f(2/\sqrt{3}) = 3.0792$

Abs. minimum at: $f(3) = -15$

[4] $f'' = 3e^x + 4\cos x - \frac{1}{2x^2}$

Then, $f' = 3e^x + 4\sin x - \left(-\frac{1}{2x}\right) + c$

$= 3e^x + 4\sin x + \frac{1}{2x} + c$, c = arbitrary constant

Integrate one more time and get,

$$\boxed{f = 3e^x - 4\cos x + \frac{1}{2}\ln|x| + cx + D}, \quad D = \text{another constant}$$

check answer: $f' = 3e^x + 4\sin x + \frac{1}{2x} + c$

$f'' = 3e^x + 4\cos x - \frac{1}{2x^2}$ ✓

[5](a) $\lim_{x \to -2} \left(\frac{x^2}{x^2-4} + \frac{1}{x+2}\right)$

$= \lim_{x \to -2} \left(\frac{x^2+x-2}{x^2-4}\right)$, by common denominator

$= \lim_{x \to -2} \frac{(x-1)\cancel{(x+2)}}{(x-2)\cancel{(x+2)}} = \lim_{x \to -2} \frac{x-1}{x-2} = \frac{-3}{-4} = \boxed{\frac{3}{4}}$

(b) $\lim_{x \to 0} \frac{\cos^2 x - 1}{x^2}$. Plugging in $x = 0$ gives $\frac{0}{0} \Rightarrow$ indeterminate

Apply L.H. Rule: $\frac{(\cos^2 x - 1)'}{(x^2)'} = \frac{-2\cos x \cdot \sin x}{2x} \xrightarrow[x=0]{\text{Plug in}} \frac{0}{0}$

Still indeterminate!

Apply L.H. again: $\dfrac{-(\cos x \cdot \sin x)'}{x'} = \dfrac{-(\cos^2 x - \sin^2 x)}{1} \xrightarrow[x=0]{\text{Plug in}} \dfrac{-1+0}{1}$

Answer: $\boxed{\lim_{x\to 0} \dfrac{\cos^2 x - 1}{x^2} = -1}$

[6] The given function is $g(x) = \int^x f(t)\,dt$, with the graph of $f(t)$ shown.

(a) $g(-1)$ is positive. Reason: $g(-1) = \int_1^{-1} f(t)\,dt = -\int_{-1}^{1} f(t)\,dt = -$(negative area) $=$ positive

$g(3)$ is positive. Reason: $\int_1^3 f(t)\,dt =$ a positive area, since $f > 0$ on the interval

$g(6)$ is positive. Reason: $\int_1^6 f(t)\,dt = \int_1^5 + \int_5^6 = \left.\begin{matrix}\text{large positive area} + \\ \text{small negative area}\end{matrix}\right\} = +$

(b) Value of $g(5) = \int_1^5 f(t)\,dt =$ area of $\Delta = \frac{1}{2} \times 4 \times 2 = 4$

$g'(5) = 0$. Reason: By the FTC, since f is continuous at $t=5$; $g'(5) = f(5)$

Answers: $g(5) = 4$, $g'(5) = 0$

(c) On $(-4, 10)$:

g is $\uparrow$ on: $(1,5)$, $(7,10)$

g is $\downarrow$ on: $(-4,1)$, $(5,7)$

Reason: $g'(x) = f(x)$, which means $g\uparrow$ when $f>0$, and $g\downarrow$ when $f<0$

(d) The critical points of $g(x)$ are at: $x = 1, 5, 7$.
The end points are at $x=-4$ and 10. We can evaluate g at all these points to get the table shown below

x	-4	1	5	7	10
$g(x)$	$2+6$	0	4	$4-1$	$3+4\cdot5$

Abs. maximum: $g(-4) = 8$

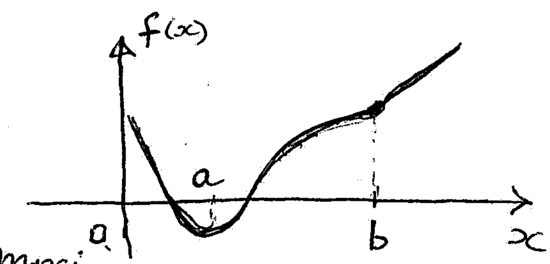
Abs. minimum: $g(1) = 0$

[7] (a) f is continuous everywhere.

Has C.P. @ $x=a$, and $f'(a)$ exists

C.P. @ $x=b$, and $f'(b)$ DNE

Has local minimum @ $x=a$. No local maximum



(b) The limit based definition of derivative of $f(x)$ at $x=a$:

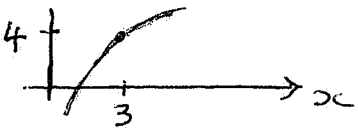
$f'(a) = \lim_{x\to a} \dfrac{f(x) - f(a)}{x-a}$ OR $f'(a) = \lim_{h\to 0} \dfrac{f(a+h) - f(a)}{h}$

If $f(x) = g(x)\cdot h(x)$, then

$f'(a) = \lim_{x\to a} \dfrac{g(x)\cdot h(x) - g(a)\cdot h(a)}{x-a} = \lim_{h\to 0} \dfrac{g(a+h)\cdot h(a+h) - g(a)\cdot h(a)}{h}$

(c) $\int_2^5 S'(x)\,dx = 300$ means: The net change in elevation between mile 2 and 5 of the trail is 300 feet. The units of $\int S'(x)\,dx$ = feet.

(d) Since $f' > 0$ and $f'' < 0$, the function is increasing and concave down. If it passes through the point $(3,4)$ it must intersect the x-axis exactly once.



$\therefore$ $f(x) = 0$ has exactly one solution.

(e) Since $e^{\sin^2 x}$ is continuous for all x, we can use the FTC.

(i) $\int_0^{\pi/2} \frac{d}{dx}\left(e^{\sin^2 x}\right) dx = e^{\sin^2 x}\Big]_0^{\pi/2} = e^1 - e^0 = \boxed{e-1}$

(ii) $\frac{d}{dx}\left(\int_0^{\pi/2} e^{\sin^2 x}\, dx\right) = \frac{d}{dx}\left[\text{constant}\right] \boxed{=0}$

The value of $\int_0^{\pi/2} f(x)\,dx$ = const.

Grading notes:

[1] 3 pt = correct hypotheses + statement of part 1 + statement of part 2
 1 pt = explain significance.

[2] 1.5 pt = simplify to powers of x; 2 pt = differentiate 1st two terms
 1.5 pt = differentiate term with P.R. (−0.5 pt for unsimplified parts)
 (b) 2+2 = correctly differentiate LHS + RHS; 1 pt = correctly solve for dy/dx

[3] 1 pt = correct y', with correct domains; 1 pt = find c.p.'s at $\pm 2/\sqrt{3}$;
 1 pt = find c.p. @ $x=0$, with steps/reasons; 1 pt = sign analysis for local extremes
 1 pt = correct strategy leading to abs. extremes.

[4] 1.5 pt = correctly integrate once (w/o constant); 1.5 pt = correctly integrate 2nd time (w/o constant); 1+1 pt = include constants & treat correctly

[5] (a) 1 pt = correctly common denom; 3 pt = correctly factor & cancel;
 1 pt = plug in & get answer [If using L.H. rule, 1 pt = check indeterminate]
 (b) 0.5 pt + 1.5 pt = check indet. + apply L.H. Rule; 0.5 + 1.5 pt = 2nd round,
 1 pt = plug in and get correct answer.

[6] (a) = (b) = (c) = (d) = 2 points each: generally 50/50 split between answer & reason

[7] 2 points for each of 5 parts. Again, generally 50/50 split between answers & reasons, though other factors considered case-by-case.