Cohomology of Topological Groups and Grothendieck Topologies

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Outline

Cohomology of Topological Groups

- There is a good theory of group cohomology for a group G and a G-module A.
- What if G is a topological group and A is a topological G-module (an abelian topological group with a continuous G-action $G \times A \to A$)?
- We want a different theory that gives more, or better, information.

Lichtenbaum's Topology

Definition (Lichtenbaum)

The Grothendieck topology T_G^L*:*

- *Category* $= \mathcal{C}_G$*: G-spaces and continuous G-maps*
- *Coverings:* $\{f_i : X_i \to X\}$ *such that for every* $x \in X$ *there is a neighborhood* U *of* x*, an index* i*, and a continuous section* $s : U \to X_i$ *with* $f_i \circ s = \mathrm{id}_U$*.*

Theorem (Lichtenbaum)

Let A *be a topological G-module and* $\tilde{A} = \mathrm{Hom}_{\mathcal{C}_G}(-, A)$*. Then* $H^n(T_G^L, pt, \tilde{A}) = H^n_{ss}(G, A)$*, Wigner's semisimplicial cohomology.*

Application: Weil-étale Cohomology

- Let K be a number field.
- Lichtenbaum "combined" many T_G^L's, where the G's are local Weil groups of K and the global Weil group of K, and constructed the Grothendieck topology T_K.
- Euler characteristics of T_K should be related to special values of ζ_K.

The Cochain Theories

The definitions will be given as the cohomologies of complexes of inhomogeneous cochains:

$$H^n_*(G, A) = H^n(C^n_*(G, A), \delta_n)_{n=0}^{\infty}$$

$$\delta_n : C^n_*(G, A) \to C^{n+1}_*(G, A)$$

$$\begin{aligned}\delta_n(f)(g_0, \ldots, g_n) &= g_0 \cdot f(g_1, \ldots, g_n) \\ &+ \sum_{k=1}^{n} (-1)^k f(g_0, \ldots, g_{k-1}g_k, \ldots, g_n) \\ &+ (-1)^{n+1} f(g_0, \ldots, g_{n-1})\end{aligned}$$

Continuous and Measurable Cochain Theories

Continuous Cochain Theory

$$C_c^n(G, A) = \{\text{continuous maps } G^n \to A\}$$

Measurable Cochain Theory

$$C_m^n(G, A) = \{\text{measurable maps } G^n \to A\}$$

Locally Continuous Theories

Locally Continuous Cochain Theory

$$C^n_{lc}(G,A) = \left\{ \begin{array}{c} \text{maps } G^n \to A \text{ continuous on a} \\ \text{neighborhood of } (e,\ldots,e) \end{array} \right\}$$

Locally Continuous Measurable Cochain Theory

$$C^n_{lcm}(G,A) = \left\{ \begin{array}{c} \text{measurable maps } G^n \to A \text{ continuous on a} \\ \text{neighborhood of } (e,\ldots,e) \end{array} \right\}$$

Reinterpretation as Cohomologies of Grothendieck Topologies

Theorem (Minevich)

There exist Grothendieck topologies $T_G^c, T_G^m, T_G^{lc}, T_G^{lcm}$ and sheaves F_A for a topological G-module A on these topologies, such that $H_*^n(G, A) \cong H^n(T_G^*, pt, F_A)$.

Description of the Grothendieck Topologies

Underlying Categories

- T_G^c: G-spaces and continuous G-maps
- T_G^m: G-spaces and measurable G-maps
- T_G^{lc}: pointed G-spaces (X, x) and G-maps $f : (X, x) \to (Y, y)$ with $f(x) = y$ that are continuous on a neighborhood of x
- T_G^{lcm}: pointed G-spaces (X, x) and measurable G-maps $f : (X, x) \to (Y, y)$ with $f(x) = y$ that are continuous on a neighborhood of x

Coverings

Single morphisms $\{f : X \to Y\}$ such that there is a section $s : Y \to X$ that is almost a morphism in the respective category: s must satisfy all properties except it need not be a G-map.

The Sheaves F_A

- For T_G^c and T_G^m, $F_A = \mathrm{Hom}(-, A)$.
- For T_G^{lc}, $F_A(X, x)$ is the set of G-maps $f : X \to A$ continuous on a neighborhood of x ($f(x)$ need not be 0).
- For T_G^{lcm}, $F_A(X, x)$ is the set of measurable G-maps $f : X \to A$ continuous on a neighborhood of x.

Morphisms of Grothendieck Topologies

Definition

A morphism of Grothendieck topologies $g : T_1 \to T_2$ *is a functor* $\mathrm{Cat}(T_1) \to \mathrm{Cat}(T_2)$ *which preserves fibered products and coverings.*

- Let $\mathcal{S}(T)$ be the category of sheaves of abelian groups on T.
- A morphism $g : T_1 \to T_2$ induces a functor $g_* : \mathcal{S}(T_2) \to \mathcal{S}(T_1)$ given by $g_*F(X) = F(gX)$.

Lemma

If for every $X \in \mathrm{Cat}(T_1)$ *and every covering* $\{Y_i \to gX\}$ *in* T_2 *there is a covering* $\{X_j \to X\}$ *in* T_1 *such that* $\{gX_j \to gX\}$ *refines* $\{Y_i \to gX\}$, *then* g_* *is exact.*

The Use of Single Morphisms

Lemma

If g_* is exact, then for every $X \in \mathrm{Cat}(T_1)$, every sheaf F on T_2, and all $n > 0$ we have $H^n(T_1, X, g_*F) = H^n(T_2, gX, F)$.

Lemma (Minevich)

Suppose that in a Grothendieck topology T, $\{f_i : X_i \to X\}$ is a covering if and only if $\{\coprod f_i : \coprod X_i \to X\}$ is a covering. Let T^1 be the topology on $\mathrm{Cat}(T)$ whose coverings are single-morphism coverings in T. Then for the morphism $g : T^1 \to T$ which is the identity on $\mathrm{Cat}(T)$, g_* is exact.

- Applies to T_G^L
- Possible application to Weil-étale cohomology

Map from $H^n_{ss}(G, A)$ to $H^n_m(G, A)$

Theorem (Minevich)

If G is second countable, there are natural maps $H^n_{ss}(G, A) \to H^n_m(G, A)$.

Proof

- Grothendieck topology $T^{L,1}_{G,sc}$
- Category: second countable G-spaces and continuous G-maps
- Coverings: single-morphism coverings in T^L_G
- Natural morphisms $T^L_G \xleftarrow{\alpha} T^{L,1}_{G,sc} \xrightarrow{\beta} T^m_G$
- α_* and β_* are exact.
- Inclusion $\alpha_* \tilde{A} \hookrightarrow \beta_* F_A$

Maps from $H^n_{ss}(G,A)$ to $H^n_{lc}(G,A)$ and $H^n_{lcm}(G,A)$

Theorem (Minevich)

There are natural maps $H^n_{ss}(G,A) \to H^n_{lc}(G,A)$. If G is second countable, there are natural maps $H^n_{ss}(G,A) \to H^n_{lcm}(G,A)$.

Proof

- Grothendieck topology $T^{L,1}_{G,sc,*}$
- Category: second countable pointed G-spaces and continuous G-maps $f : (X,x) \to (Y,y)$ with $f(x) = y$
- Coverings: single-morphism coverings in T^L_G
- Natural morphisms $T^L_G \xleftarrow{\alpha} T^{L,1}_{G,sc,*} \xrightarrow{\beta} T^{lcm}_G$
- α_* and β_* are exact.
- Inclusion $\alpha_*\tilde{A} \hookrightarrow \beta_* F_A$

Statement of the Theorem

- $\mathcal{M}_G$ is the category of topological G-modules
- $\mathcal{M}_G^{pm}$ is the category of pseudometrizable topological G-modules
- $\mathcal{M}_G^{cm}$ is the category of completely metrizable topological G-modules

Theorem (L. Brown - Minevich)

If G is "weakly separable" and $A, B \in \mathcal{M}_G^{cm}$ then for all n

$$\mathrm{Ext}^n_{\mathcal{M}_G}(A, B) \cong \mathrm{Ext}^n_{\mathcal{M}_G^{pm}}(A, B) \cong \mathrm{Ext}^n_{\mathcal{M}_G^{cm}}(A, B)$$

- We can define $H^n(G, A) = \mathrm{Ext}^n_{\mathcal{M}_G}(\mathbb{Z}, A)$.
- For completely metrizable A, we can work in $\mathcal{M}_G^{cm}$ instead.

Definitions for Topological Spaces

Definition

A topological space X is pseudometrizable if its topology is induced by a pseudometric, i.e. a "metric" d for which we could have $d(x, y) = 0$ without $x = y$.
X is completely metrizable if its topology is induced by a metric d under which X is complete.

Definition

A topological group G is weakly separable if for every open set U in G, the covering $\{xU\}_{x \in G}$ has a countable subcovering.

Proper Morphisms and Exact Sequences

Definition

A morphism $f : A \to B$ in an additive category is proper if its image is well-defined, i.e. $\mathrm{Coker}(\ker f) \xrightarrow{\sim} \ker(\mathrm{Coker}\, f)$.

- In the categories $\mathcal{M}_G, \mathcal{M}_G^{pm}$, and $\mathcal{M}_G^{cm}$, $f : A \to B$ is proper iff it is open as a map onto its image.
- An injection $A \hookrightarrow B$ is proper iff it is a homeomorphism onto its image.

Definition

A sequence $A \xrightarrow{f} B \xrightarrow{g} C$ is exact if f and g are proper, $g \circ f = 0$, and $\mathrm{Im}(f) \xrightarrow{\sim} \ker g$.

Definition of Yoneda Ext

$$\mathrm{Ext}^n(A, B) = \{\text{exact sequences } 0 \to B \to E_n \to \cdots \to E_1 \to A \to 0\}/ \sim$$

where $\sim$ is the equivalence relation generated by commutative diagrams

$$\begin{array}{ccccccccccccc} 0 & \to & B & \to & E_n & \to & \cdots & \to & E_1 & \to & A & \to & 0 \\ & & \| & & \downarrow & & & & \downarrow & & \| & & \\ 0 & \to & B & \to & E'_n & \to & \cdots & \to & E'_1 & \to & A & \to & 0 \end{array}$$

The Proof

- The hard part is showing $\mathrm{Ext}^n_{\mathcal{M}_G}(A, B) = \mathrm{Ext}^n_{\mathcal{M}_G^{pm}}(A, B)$.
- The key is in showing that, if we have a proper injection $i : B \hookrightarrow E$ where B is a pseudometrizable G-module and E is any topological G-module, then there is a coarser topology on E with which i is still a proper injection and E is a pseudometrizable topological G-module.
- This uses the fact that a topological group is pseudometrizable iff it is first countable
- We construct a good basis for E at 0 using the balls of radius $1/n$ in B