Synthetic Cevian Geometry

Igor Minevich
(Joint Work with Patrick Morton)

February 25, 2016

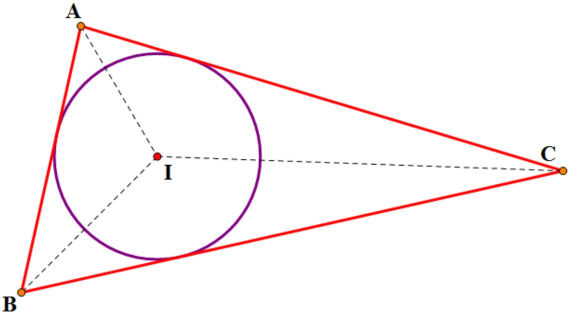
Outline

Part I

Almost 10,000 Interesting Points!

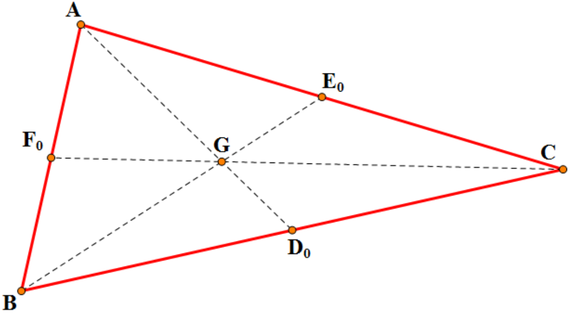
Incenter $I = X(1)$

- The center of the circle inscribed in the triangle
- The intersection of the angle bisectors



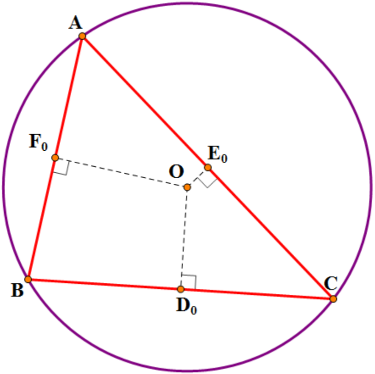
Centroid $G = X(2)$

- The intersection of the medians AD_0, BE_0, CF_0



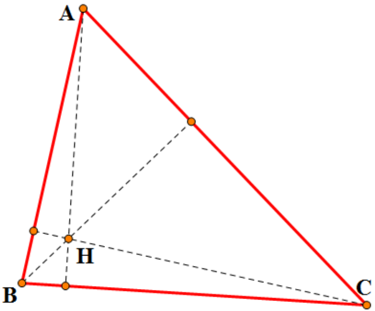
Circumcenter $O = X(3)$

- The center of the circle through the vertices A, B, C
- The intersection of the perpendicular bisectors of the sides



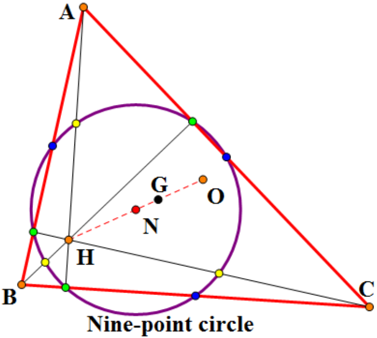
Orthocenter $H = X(4)$

- The intersection of the altitudes



Nine-Point Center $N = X(5)$

- The midpoint of OH
- The center of the nine-point circle

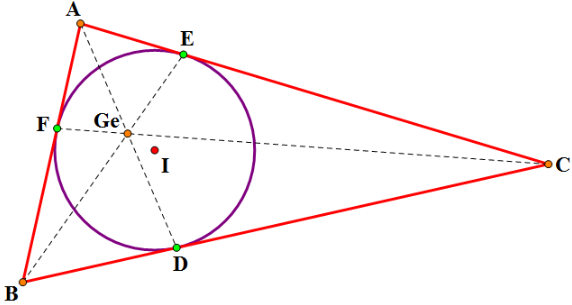


The 9-point circle goes through:

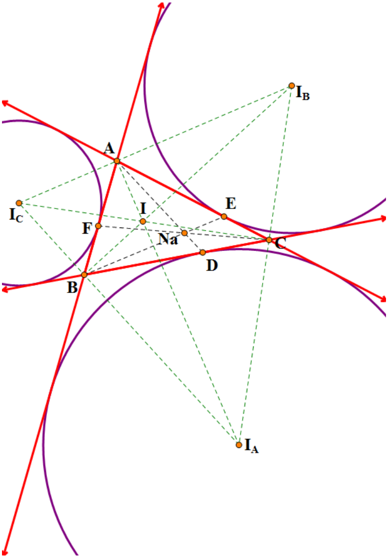
- the feet of the altitudes
- the midpoints of the sides
- the midpoints of AH, BH, CH

Gergonne Point $Ge = X(7)$

If D, E, F are the places where the incircle touches the sides, then Ge is the intersection of AD, BE, CF



Nagel Point $Na = X(8)$



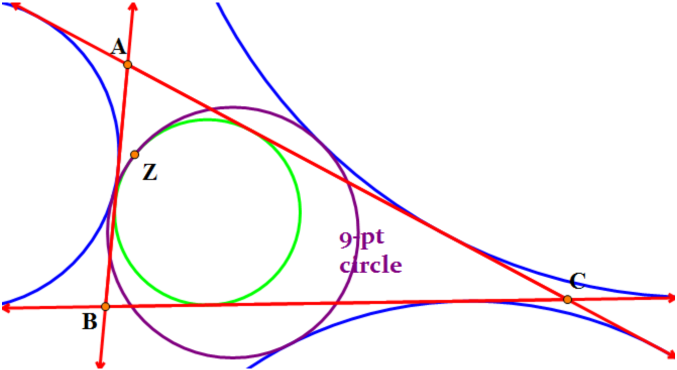
- If D, E, F are the places where the excircles touch the sides, then Na is the intersection of AD, BE, CF.
- D, E, F are precisely the places you'll get to if you walk around the perimeter of the triangle halfway from each of the vertices.

Feuerbach Point $Z = X(11)$

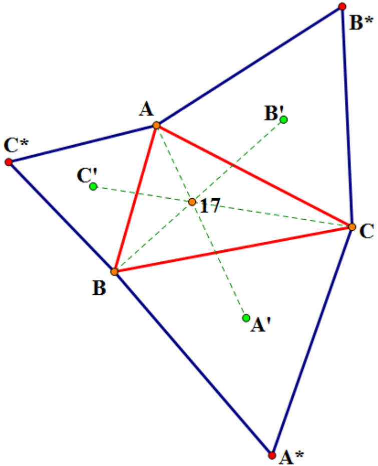
Theorem (Feuerbach)

The nine-point circle is tangent to the incircle and the three excircles.

The point of tangency with the incircle is called the Feuerbach Point Z

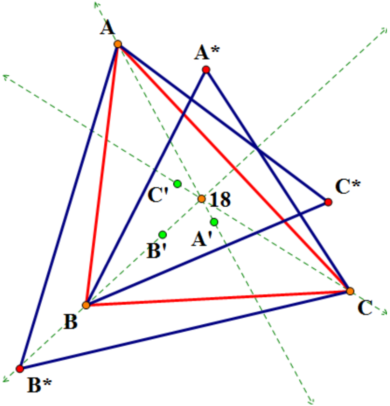


Just For Fun: 1st Napoleon Center $X(17)$



Draw equilateral triangles A^*BC, AB^*C, ABC^*outwards on the sides of the triangle, then connect the centers A', B', C' of the equilateral triangles to the opposite vertices of the triangle. $X(17)$ is the intersection of these lines.

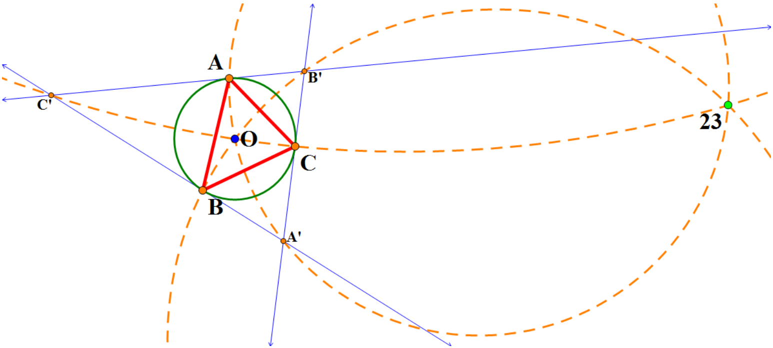
Just For Fun: 2nd Napoleon Center $X(18)$



Draw equilateral triangles A^*BC, AB^*C, ABC^* inwards on the sides of the triangle, then connect the centers A', B', C' of the equilateral triangles to the opposite vertices of the triangle. $X(18)$ is the intersection of these lines.

Just For Fun: Far-Out Point $X(23)$

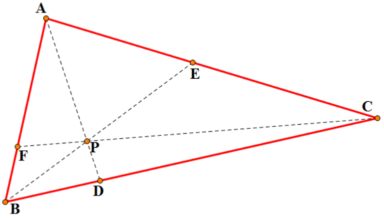
Let $A'B'C'$ be the triangle whose sides are tangent to the circumcircle of ABC. The circles through AOA', BOB', and COC' all go through one point: $X(23)$.



Part II

Some Theorems from Projective Geometry

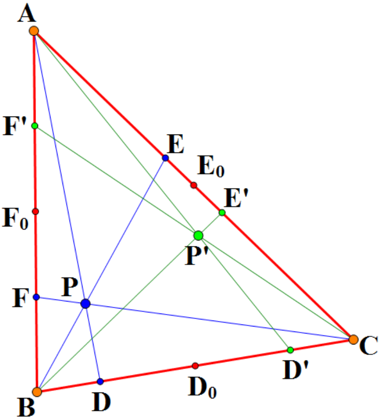
Čeva's Theorem: Why The Maps Are Well-Defined



Let D, E, F be points on the lines BC, CA, AB, respectively. Then AD, BE, CF are concurrent (i.e. intersect in one point P) if and only if

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1$$

Isotomic Conjugate Map ι: Definition



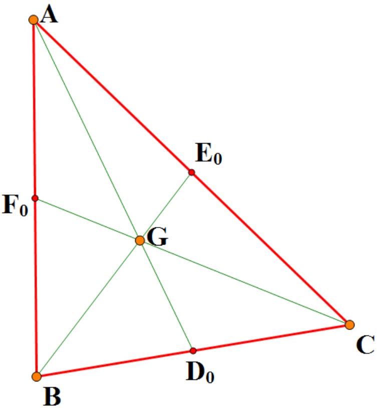
Let P be a point not on the sides of ABC. Let D, E, F be its traces:

$$D = AP \cap BC$$
$$E = BP \cap AC$$
$$F = CP \cap AB$$

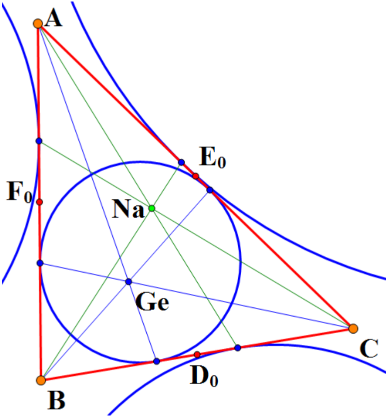
Let D', E', F' be the reflections of D, E, F across the midpoints of the sides. Then AD', BE', CF' are concurrent at the point $P' = \iota(P)$, the isotomic conjugate of P, and $P = \iota(P')$.

Isotomic Conjugate Map ι: Examples

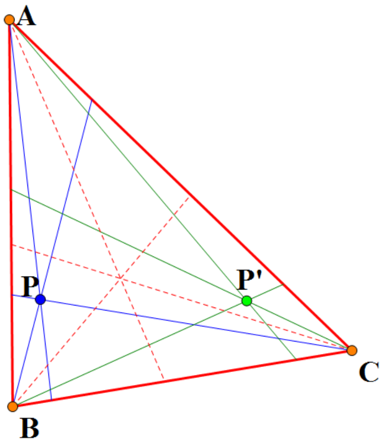
The centroid G is a fixed point:
$\iota(G) = G$.



$\iota(Ge) = Na$.



Isogonal Conjugate Map γ: Definition

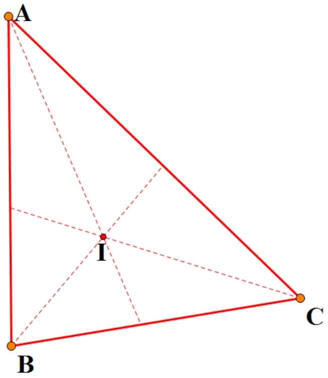


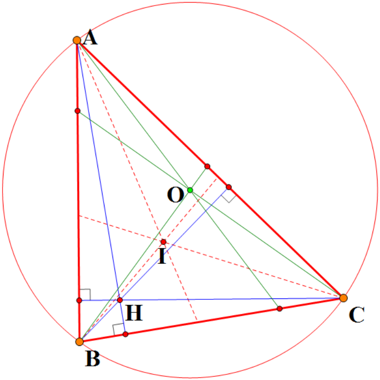
Let P be a point not on the sides of ABC. Reflect the lines AP, BP, CP across the angle bisectors (dashed red lines). The three resulting lines are concurrent at the point $P' = \gamma(P)$, the isogonal conjugate of P, and $\gamma(P') = P$.

Isogonal Conjugate Map γ: Examples

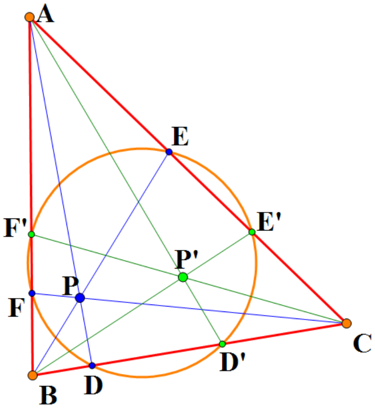
The incenter I is a fixed point:
$\gamma(I) = I$.

$\gamma(O) = H$.





Cyclocevian Conjugate Map ϕ: Definition



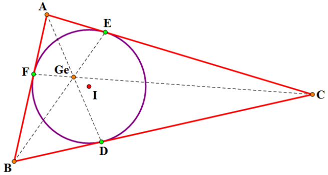
Let D, E, F be the traces of P:

$$D = AP \cap BC$$
$$E = BP \cap AC$$
$$F = CP \cap AB$$

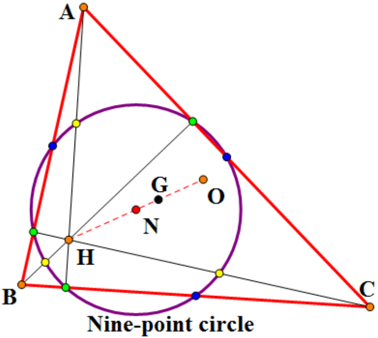
Let D', E', F' be the other intersections of the circle through D, E, F with the sides of the triangle. Then AD', BE', CF' are concurrent at the point $P' = \phi(P)$, the cyclocevian conjugate of P, and $P = \phi(P')$.

Cyclocevian Conjugate Map ϕ: Examples

The Gergonne Point is a fixed point: $\phi(Ge) = Ge$.

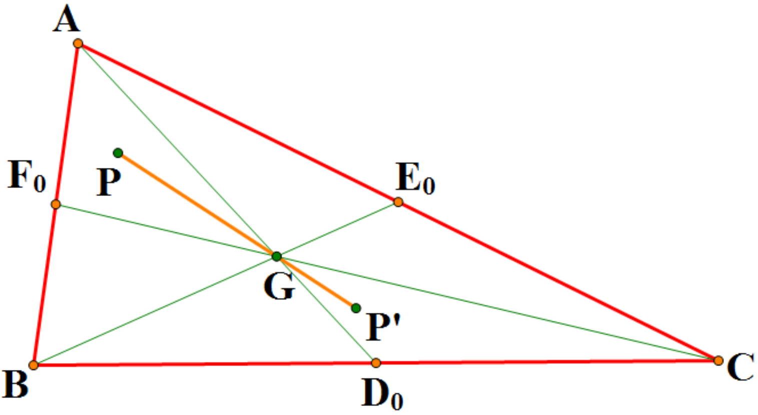


$\phi(H) = G$.



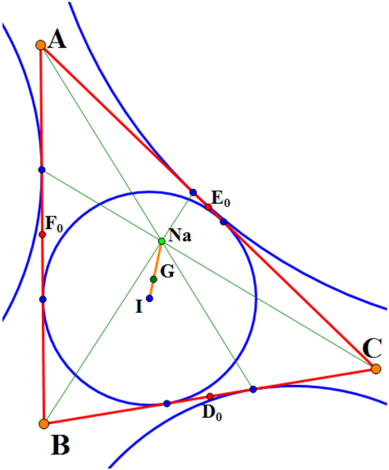
Complement Map K: Definition

$K(P) = P'$ where $PG = -2GP'$. In this case, $K^{-1}(P') = P$.
$K(G) = G$; $K(A) = D_0$.

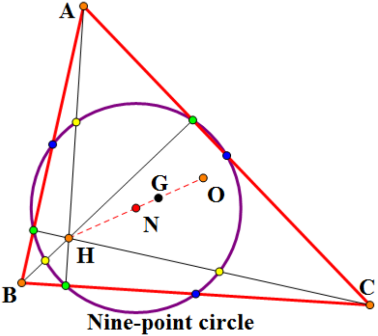


Complement Map K: Examples

$K(Na) = I.$



$K(H) = O$ and $K(O) = N.$



The Theorem

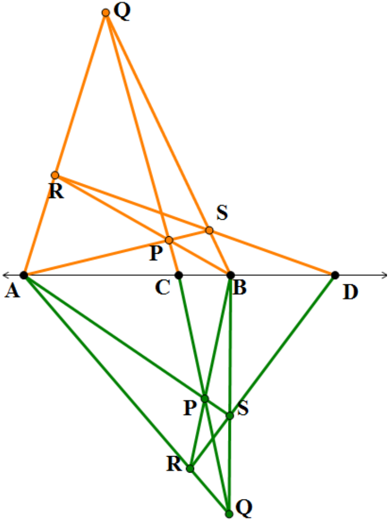
Theorem (D. Grinberg, 2003)

$$\phi = \iota \circ K^{-1} \circ \gamma \circ K \circ \iota$$

Part III

Some Theorems from Projective Geometry

Harmonic Conjugates



Let A, B, C lie on one line l. The harmonic conjugate of C with respect to A and B is the point D on l such that the cross ratio equals -1:

$$\frac{AC/CB}{AD/DB} = -1$$

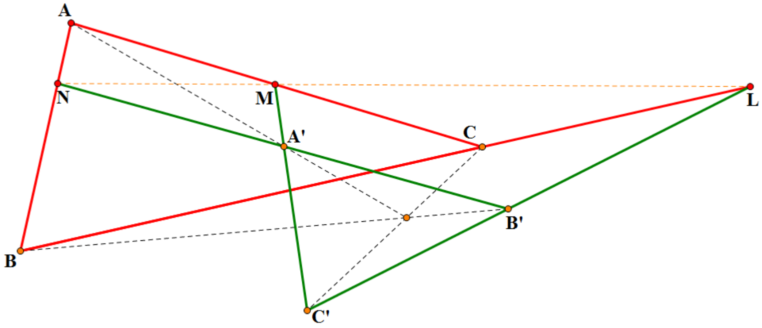
Take any triangle PQR such that A is on QR, B is on PR, and C is on PQ. Construct $S = AP \cap BQ$. Then $D = RS \cap l$.

Desargues' Theorem

Let ΔABC and $\Delta A'B'C'$ be two triangles. Then AA', BB', CC' are concurrent if and only if

$$L = BC \cap B'C', M = AC \cap A'C', \text{ and } N = AB \cap A'B'$$

are collinear.

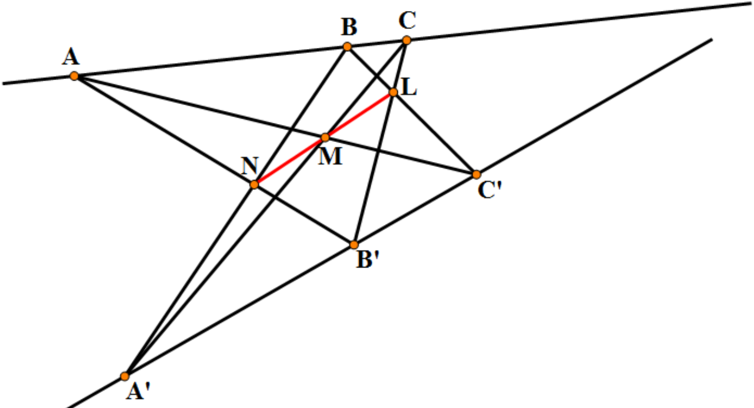


Pappus's Theorem

Let A, B, C be collinear and A', B', C' be collinear. Then the cross-joins

$$L = BC' \cap B'C, M = AC' \cap A'C, \text{ and } N = AB' \cap A'B$$

are collinear.

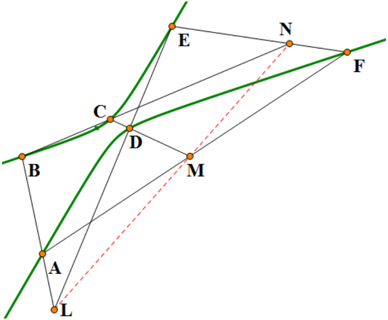


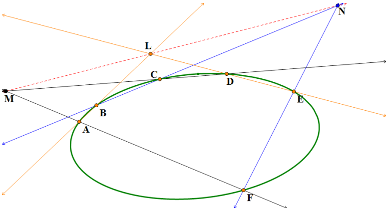
Pascal's "Mystical Hexagram" Theorem

Let A, B, C, D, E, F be points on a conic (parabola, hyperbola, or ellipse). Then the intersections of the "opposite" sides, namely

$$AB \cap DE, BC \cap EF, \text{ and } CD \cap AF$$

are collinear.

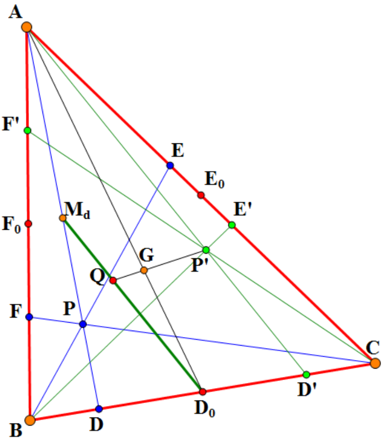




Part IV

The Synthetic Proof of Grinberg's Theorem

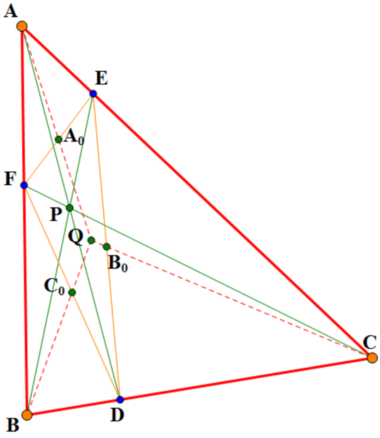
Lemma 1



Lemma (Grinberg)

Let ABC be a triangle and D, E, F the traces of point P. Let D_0, E_0, F_0 be the midpoints of BC, CA, AB, and let M_d, M_e, M_f be the midpoints of AD, BE, CF. Then D_0M_d, E_0M_e, F_0M_f meet at the isotomcomplement $Q = K \circ \iota(P)$ of P.

Lemma 2 (The Key)



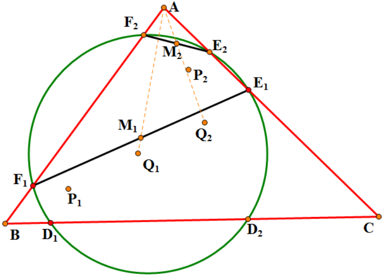
Key Lemma (Grinberg)

Let ABC be a triangle and D, E, F the traces of point P. Let A_0, B_0, C_0 be the midpoints of EF, FD, DE. Then AA_0, BB_0, CC_0 meet at the isotomcomplement $Q = K \circ \iota(P)$ of P.

We proved this lemma synthetically, using projective geometry to reduce it to the previous lemma.

Proof of the Theorem

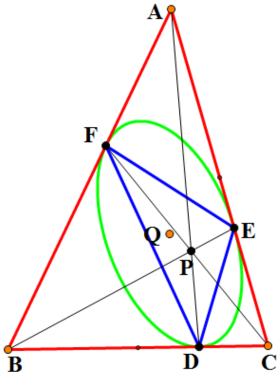
- $\phi(P) = \iota \circ K^{-1} \circ \gamma \circ K \circ \iota(P)$ if and only if $(K \circ \iota)(\phi(P)) = \gamma((K \circ \iota)(P))$
- Enough to prove that if $P_2 = \phi(P_1)$ then the isotomcomplements of P_1 and P_2 are isogonal conjugates, which is not very hard.

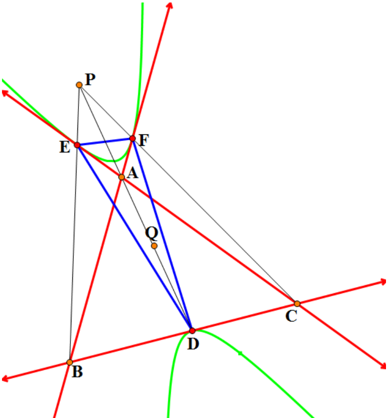


Center of the Inconic

Theorem (Grinberg)

The isotomcomplement is also the center of the inconic, the unique conic that is tangent to the sides of ABC at the points D, E, F!





Some of Our Results

Real Projective Plane $\mathbb{RP}^2$

- Embed $\mathbb{R}^2$ in the real projective plane $\mathbb{RP}^2$ by adding a "line at infinity," l_∞
- "Points at ∞" can be thought of as directions of lines. Two lines go through the same point at ∞ iff they are parallel.
- The line at infinity, l_∞, is the set of all points at ∞.
- Now any two lines intersect and any two points are joined by a line (self-dual).
- All projective geometry theorems work in this context.

Automorphisms of $\mathbb{RP}^2$

- An automorphism of $\mathbb{RP}^2$ α (also called a projective collineation) is a map from $\mathbb{RP}^2$ to $\mathbb{RP}^2$ that takes points to points and lines to lines that preserves incidence
- If a point P lies on line l, then $\alpha(P)$ lies on $\alpha(l)$.
- Given any four points A, B, C, D (no 3 collinear) and any other four points A', B', C', D' (no 3 collinear), there is a unique automorphism taking $A \mapsto A', B \mapsto B', C \mapsto C', D \mapsto D'$.

Affine Maps

Definition

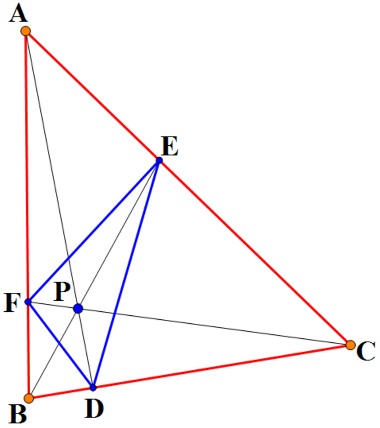
An automorphism of $\mathbb{RP}^2$ is called an affine map if it takes l_∞ to itself.

- Parallel lines go to parallel lines.
- An affine map is uniquely by determined by the images of three points A, B, C.
- An affine map preserves ratios along lines.

Affine Maps: Examples

- Example: rotations (rotate around a point)
- Example: translations (shift up/down/left/right/etc.)
- Example: dilatations (dilate everything from some center)
- Example: reflections (reflect about some line)
- Example: the complement map K
- NOT ι, γ, or ϕ.

T_P

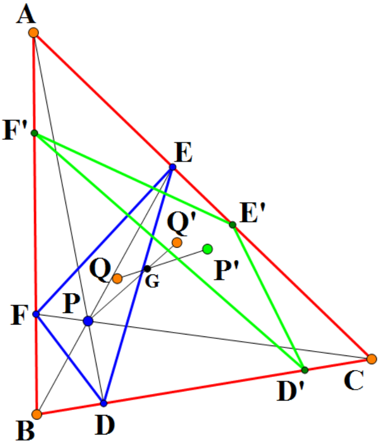


An affine map is uniquely determined by the images of three points A, B, C.

Definition

If D, E, F are the traces of P, T_P is the unique affine map taking ABC to DEF.

Our Notation



- DEF is the cevian triangle of P with respect to ABC.
- $D'E'F'$ is the cevian triangle of P' with respect to ABC.
- $P' = \iota(P)$.
- $Q' = K(P) =$ isotomcomplement of P'.
- $Q = K(\iota(P)) =$ isotomcomplement of P.

Fixed Points

Theorem (Ehrmann, Morton, —)

If $\iota(P)$ is a finite point, then the isotomcomplement of P is the unique fixed point of T_P.

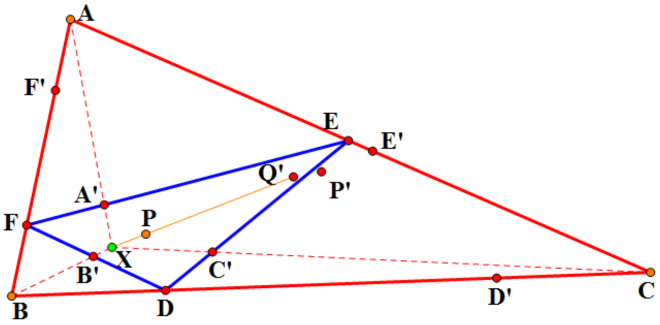
Theorem (Morton, —)

$(T_P \circ K)(P) = P.$

$T_P \circ T_{P'}$

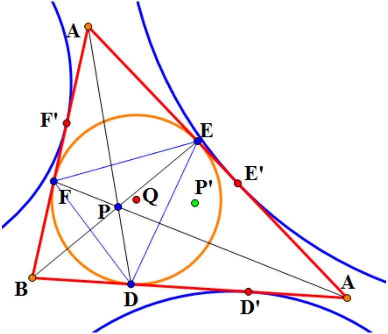
Theorem

$T_P \circ T_{P'}$ is either a translation or a dilatation. Let $D'E'F'$ be the cevian triangle of P' and $A' = T_P(D'), B' = T_P(E'), C' = T_P(F')$. Let $Q' = K(P)$. Then AA', BB', CC', and PQ' are concurrent at the fixed point of $T_P \circ T_{P'}$.



$P = Ge$

c



Let $P = Ge$. Then:

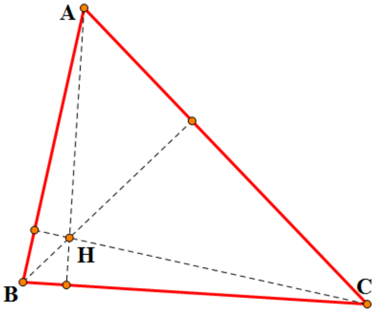
- $Q = I$, the incenter.
- $P' = Na$, the Nagel point.
- D, E, F are the points of contact of the incircle with the sides of ABC.
- $QD \perp BC$
- $QE \perp AC$
- $QF \perp AB$

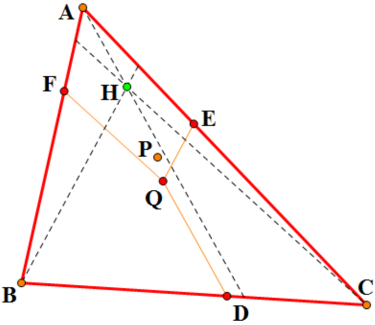
Generalizing

- P is the generalized Gergonne point.
- P' is the generalized Nagel point.
- Q is the generalized incenter.
- What if we "generalize perpendicularity" by saying "$QD \perp BC$," "$QE \perp AC$", and "$QF \perp AB$"?

Generalized Orthocenter

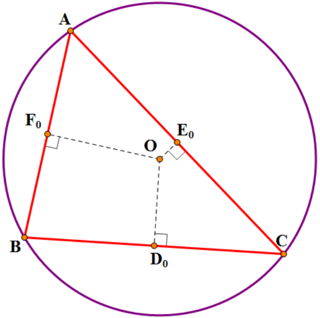
- "$QD \perp BC$," "$QE \perp AC$", and "$QF \perp AB$".
- The lines through A, B, C parallel to QD, QE, QF, resp. are concurrent at the generalized orthocenter H.

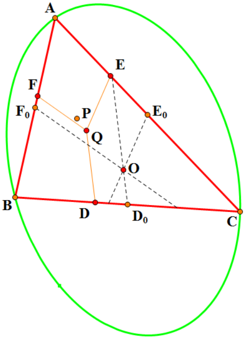




Generalized Circumcenter

- "$QD \perp BC$," "$QE \perp AC$", and "$QF \perp AB$".
- The lines through D_0, E_0, F_0 parallel to QD, QE, QF, resp. are concurrent at the generalized circumcenter O.

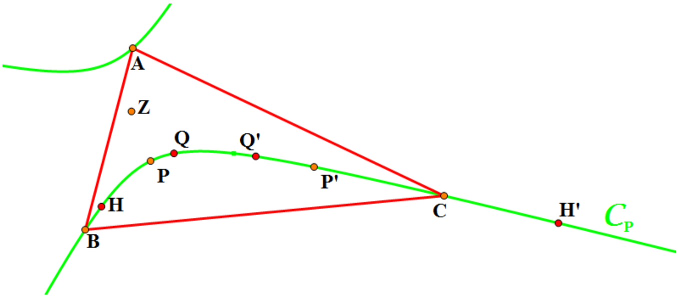




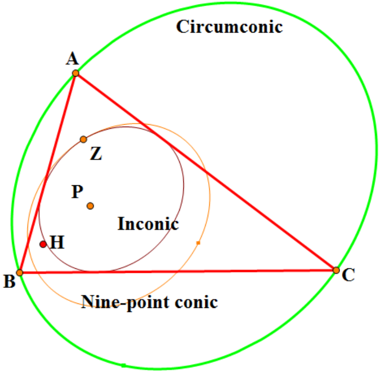
The Conic $\mathcal{C}_P$

Theorem (Morton, —)

The points A, B, C, P, $P' = \iota(P)$, $Q = K \circ \iota(P)$, $Q' = K(P)$, H, and $H' = H(P')$ all lie on one conic $\mathcal{C}_P$.



Three Conics



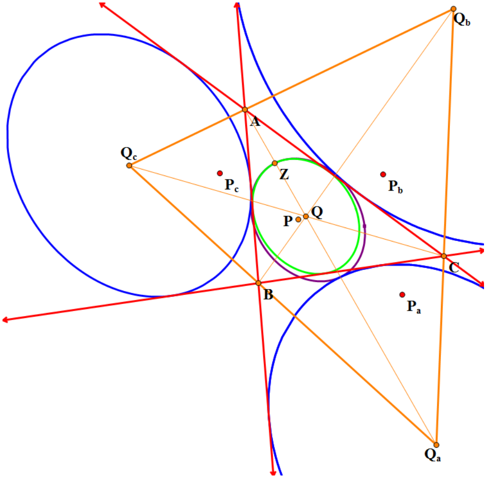
Theorem (Morton, —)

The natural generalizations of the incircle, circumcircle, and nine-point circle are conics that "point in the same direction."

Theorem (Morton, —)

The map $T_P K^{-1} T_{P'}$ is a dilatation or a translation that takes the circumconic to the inconic.

Generalized Feuerbach Theorem



Theorem (Morton, —)

The generalization $\mathcal{N}$ of the nine-point circle is tangent to the inconic at Z, the center of $\mathcal{C}_P$. The map $T_P K^{-1} T_{P'} K^{-1}$ takes the $\mathcal{N}$ to the inconic and fixes Z. There are also three exconics associated to P which are tangent to $\mathcal{N}$.