Bloch-Kato and Other Conjectures

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Abstract

We will talk about some of the history behind the celebrated Bloch-Kato conjectures and develop the terminology necessary to state the most basic conjecture, about the order of the zero of the L-function of a pure geometric p-adic representation of the absolute galois group of a number field at $s = 0$. We will also talk about related conjectures, especially that of Birch and Swinnerton-Dyer and, if time permits, the other Bloch-Kato conjecture and motivic cohomology.

Recall the **Dedekind zeta-function**:

$$\zeta_K(s) = \sum_{\mathfrak{a}} \frac{1}{(N\mathfrak{a})^s} = \prod_{\mathfrak{p}} \frac{1}{1-(N\mathfrak{p})^{-s}}$$

First, in the 19th century, it was found that

Theorem. *We have $\zeta_K(s) \sim c_K(s-1)^{-1}$ as $s \to 1$, where*

$$c_K = \frac{2^{r_1}(2\pi)^{r_2}}{\sqrt{|d_K|}}\frac{hR}{w}$$

where d_K is the discriminant of K, $w = |(\mathcal{O}_K^*)_{\mathrm{tor}}|$ is the number of roots of unity in F, h is the class number of K, and R is the regulator of K.

Hecke discovered the functional equation for ζ_K in the 1930's. This can be used to show:

Theorem. *We have $\zeta_F \sim c_K s^{r_1+r_2-1}$ as $s \to 0$, with $c_K = -hR/w$.*

Also, the functional equation can be used to find the order of zeroes at $-n$ of ζ_K, $n \in \mathbb{N}$.

Next came Birch and Swinnerton-Dyer in the 1960's:

(a) The order of the zero of $L(E, s)$ at $s = 1$ equals the rank r of $E(\mathbb{Q})$

(b) $|\text{Ш}_{E/\mathbb{Q}}| < \infty$ (recall $\text{Ш}_{E/\mathbb{Q}} = \bigcap_v \ker(H^1(G_K, E) \to H^1(G_{K_v}, E))$).

(c) $L(E, s) \sim c_E(s-1)^r$ as $s \to 1$, with

$$c_E = \frac{2^r |\text{Ш}_{E/\mathbb{Q}}| |\det h|}{|E(\mathbb{Q})_{\text{tor}}|^2} \Omega_E$$

where Ш is the Tate-Shafarevich group, h is the height pairing, and Ω_E is the real period.

Then, work on K-theory (by Grothendieck, Bass, Milnor, Gerstein, and Quillen) allowed Borel to invent his regulator and state a conjecture about the residues at $-n$.

Deligne made a conjecture regarding the residues of the L function of a number field, to multiplication by a rational number, at "critique" integers. An integer n is "critique" if the Γ-functions in the functional equation don't have a pole at n or at the point corresponding to n in the equation. This corresponds to there being no "regulator" term in the answer, so no non-torsion motivic cohomology (K-theory) around. Then Beilinson (1984) [Beĭ84] generalized the conjecture to also include a conjecture on the order of the zero of the L-function at $s = 0$, by using K-theory to work with 'critique' points. He invented his own regulator, which maps algebraic K-theory of algebraic varieties over $\mathbb{R}$ to real Deligne cohomology. It was much later proved (2002) that the Borel regulator is twice the Beilinson regulator. Then, Bloch and Kato [BK90] (1990) made a conjecture which nails down the correct formula up to sign, which works for all positive numbers. Finally, Fontaine and Perrin-Riou (1994) [FPR94] made a conjecture giving the order of the zero and the residue at all integers, up to sign. It is unclear whether or not Fontaine and Perrin-Riou's conjecture is actually a generalization of Bloch and Kato's, since the relationship of the local factors at places $v \mid p$ to the original Bloch-Kato conjecture is not clear.

The rest of this talk is a summary of Bellaïche's notes [Bel09], for which I am extremely grateful.

In this talk, I will only discuss the part of the Bloch-Kato conjecture that has to do with the order of the zero of the L-function at $s = 0$. Throughout this talk, V will be a geometric p-adic representation of G_K for K a number field (so we have a continuous linear map $G_K \xrightarrow{\rho} \mathrm{Aut}(V)$, where V is a $\mathbb{Q}_p$-vector space).

Definition 1. *An l-adic representation V is* **geometric** *if it is semisimple and $V = \bigoplus_{w\in\mathbb{Z}} V_w$, where V_w is pure of weight w.*

Definition 2. *An l-adic representation V of G_K is* **pure of weight** *w if there is a finite set Σ of places v of K such that, for all $v \notin \Sigma$, V is unramified and the characteristic polynomial P_v of Frobenius at v has roots in $\bar{\mathbb{Q}}$ with complex absolute values $q_v^{w/2}$.*

Recall a representation V is **unramified** at a place v of K if the inertia group $I(v)$ acts trivially on V ($I(v) \subseteq \ker(\rho)$).

A geometric representation is unramified almost everywhere, and de Rham at all places $v \mid p$.

We will not be discussing what it means to be de Rham, crystalline, etc. You can read about it in [FO].

Conjecture 1 (Fontaine-Mazur)**.** *Every geometric representation is isomorphic to a subquotient of $H^i_{\text{ét}}(X, \mathbb{Q}_p)(n)$ for a proper smooth variety X over K.*

Conjecture 2 (Langlands-Fontaine-Mazur)**.** *Every geometric irreducible p-adic representation of G_K is automorphic.*

Conjecture 3 (Bloch-Kato)**.** *If V is pure then*

$$\dim H^1_f(G_K, V^*(1)) - \dim H^0(G_K, V^*(1)) = \mathrm{ord}_{s=0} L(V, s) \tag{1}$$

First we will define H^1_f, then the L-function. To define the global H^1_f, we need to first define the local H^1_f's.

1 Definition of the H^1_f's

Let K be a finite extension of $\mathbb{Q}_l$.

Definition 3. *For* $l \neq p, H^1_f(G_K, V) := H^1_{\mathrm{ur}}(G_K, V) := \ker(H^1(G_K, V) \to H^1(I_K, V))$. *For* $l = p, H^1_f(G_K, V) := \ker(H^1(G_K, V) \to H^1(G_K, V \otimes_{\mathbb{Q}_p} B_{\mathrm{crys}}))$.

The 'f' for the $l = p$ case is the letter between 'e' (exponential) and 'g' (geometric):
$H^1_e(G_K, V) = \ker(H^1(G_K, V) \to H^1(G_K, V \otimes B_{\mathrm{crys}}^{\phi=1}))$ (where ϕ is the Crystalline Frobenius)
$H^1_g(G_K, V) = \ker(H^1(G_K, V) \to H^1(G_K, V \otimes B_{\mathrm{dR}}))$
and $H^1_e \subseteq H^1_f \subseteq H^1_g$, and dimension-wise the analogies are vertically as follows:

$l \neq p$			0	$\subseteq$	$H^1_{\mathrm{ur}}(G_K, V)$	$\subseteq$	$H^1(G_K, V)$		
$l = p$	0	$\subseteq$	$H^1_e(G_k, V)$	$\subseteq$	$H^1_f(G_K, V)$	$\subseteq$	$H^1_g(G_K, V)$	$\subseteq$	$H^1(G_K, V)$

There is a short exact sequence

$$0 \to D_{\mathrm{crys}}(V)^{\phi=1}/V^{G_K} \to D_{\mathrm{dR}}(V)/D^+_{\mathrm{dR}}(V) \xrightarrow{e} H^1_e(G_K, V) \to 0$$

and the map e is the **Bloch-Kato exponential** because, in the case where V is the Tate module for an abelian variety A over K, it can be identified with (the tensorization with $\mathbb{Q}_p$ of) the exponential map from an open subgroup of the Lie algebra of A to $A(K)$.

Proposition 1. *(1)* $H^i(G_K, V) = 0$ *for* $i > 2$.

(2) $H^i(G_K, V) \times H^{2-i}(G_K, V^*(1)) \xrightarrow{\cup} H^2(G_K, \mathbb{Q}_p(1)) = \mathbb{Q}_p$ *is a perfect pairing.*

(3) $\dim H^0(G_K, V) - \dim H^1(G_K, V) + \dim H^2(G_K, V) = 0$ *if* $l \neq p$ *and* $[K : \mathbb{Q}_p] \dim V$ *if* $l = p$.

Under the duality, $H^1_f(G_K, V)^\perp = H^1_f(G_K, V^*(1))$, for any l, $H^1_e(G_K, V)^\perp = H^1_g(G_K, V^*(1))$, and $H^1_g(G_K, V)^\perp = H^1_e(G_K, V^*(1))$.

Definition 4. *Let K be a number field. $H^1_f(G_K, V) = \{x \in H^1(G_K, V) \mid x_v \in H^1_f(G(v), V)$ for all $v \nmid \infty$.*

Here are two examples which show H^1_f is an interesting object to study. We can construct local Kummer morphisms for commutative group schemes over the local fields and glue them to get $\mathcal{O}_K^* \otimes_{\mathbb{Z}_p} \mathbb{Q}_p(1) \cong H^1_f(G_K, \mathbb{Q}_p(1))$. For an elliptic curve E over a number field K, $H^1_f(G_K, V_p(E)) \cong \mathrm{Sel}_p(E)$, where $V_p(E) = \varprojlim A[p^n](\bar{K}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is the Tate module of the elliptic curve and $\mathrm{Sel}_p(E) = \{x \in H^1(G_K, V_p(E)) \mid x_v \in \kappa_v(E(K_v))$. So we have an injection $E(K) \otimes_{\mathbb{Z}} \mathbb{Q}_p \hookrightarrow H^1(G_K, V_p(E))$ which is an isomorphism iff $|\text{Ш}(E)[p^\infty]| < \infty$.

2 L-Functions

Fix an embedding $\mathbb{Q}_p \xrightarrow{\iota} \mathbb{C}$.

Definition 5. *The Euler factor for $v \nmid p, \infty$ as $L_v(V, s) := \det((\mathrm{Frob}_v^{-1} q_v^{-s} - \mathrm{id})|V^{I(v)})^{-1}$, where q_v is the order of the residue field at v and Frob_v has complex entries via ι.*

The Euler factor $v \mid p$ is defined using the theory of crystalline representations.

Definition 6. *$L(V, s) = \prod_{v \nmid \infty} L_v(V, s)$.*

Theorem. *$L(V, s)$ is a well-defined meromorphic function with no zeroes on the half-plane $\Re s > w/2 + 1$ for any pure representation V of weight $w \in \mathbb{Z}$.*

Conjecture 4. *For a pure representation V of weight w, $L(V, s)$ has a meromorphic continuation to $\mathbb{C}$ and no zeroes with real part* **at least** *$w/2 + 1$. If V is irreducible, $L(V, s)$ has no poles except if $V \cong \mathbb{Q}_p(n)$, in which case there is a unique pole at $s = n + 1$, which is simple.*

This is known if V is automorphic, and it is expected we will need to prove every geometric representation is automorphic before we can prove this result.

3 More on the Bloch-Kato Conjecture

e.g. $V = \mathbb{Q}_p$: this says $\mathrm{rk}_{\mathbb{Z}}\, \mathcal{O}_K^* = r_1 + r_2 - 1$ (Dirichlet's Unit Theorem)
e.g. $V = V_p(E)$, the Tate module of an elliptic curve E/K. $V^*(1) \cong V$ by the Weil pairing, so this says $\dim H^1_f(G_K, V_p(E)) = \mathrm{ord}_{s=1} L(E,s)$. But we know $\dim H^1_f \geq \mathrm{rk}\, E(K)$, with equality iff $|\text{Ш}(E)[p^\infty]| < \infty$. So BSD implies Bloch-Kato in this case and, if $|\text{Ш}(E)[p^\infty]| < \infty$ then BSD is equivalent to Bloch-Kato.

The H^0 term is 0 unless V contains $\mathbb{Q}_p(1)$. It accounts for the pole predicted by the conjecture about $L(V,s)$.
One can define a function $L_\infty(V,s)$ and then set $\Lambda(V,s) := L(V,s)L_\infty(V,s)$.
e.g. $\Lambda(\mathbb{Q}_p, s) = \zeta_K(s)\Gamma_{\mathbb{R}}(s)^{r_1}\Gamma_{\mathbb{C}}(s)^{r_2}$, where

$$\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2), \qquad \Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$$

and r_1 = number of real places, r_2 = number of complex places.

Conjecture 5. $\Lambda(V^*(1), -s) = AB^s\Lambda(V,s)$*, where* A, B *depend on* V*.*

This is known for automorphic representations.

The way one tries to approach this conjecture is to split it up into $\geq$ and $\leq$ in equation 1. To prove $\geq$, it is a matter of constructing extensions of $\mathbb{Q}_p$ by $V^*(1)$ that land in H^1_f, i.e. nontrivial extensions between Galois representations with prescribed local properties. To prove $\leq$, one has the following theorem:

Theorem. *Under some assumptions, which are true for most representations attached to modular forms, if* L *has an analytic continuation and* Λ *satisfies the functional equation, then the Bloch-Kato conjecture for* V *is equivalent to the conjecture for* $V^*(1)$*, which implies* $\leq$ *if* V *is pure of weight* $w \neq -1$*.*

4 Motives and the Other Bloch-Kato Conjecture

There is another Bloch-Kato conjecture, about mixed motives, which is now proven ([Bel09] discusses both of these conjectures). For any field K, let

VPS_K be the category of varieties that are projective and smooth over K. There are numerous cohomlogy theories for this category, including étale cohomology, de Rham cohomology, Betti cohomology, and crystalline cohomology. Grothendieck conjectured that there is a $\mathbb{Q}$-linear, abelian, graded, semisimple category $\mathcal{M}_K$ of "pure iso-motives" (or just motives) and contravariant functors $H^i : VPS_K \to \mathcal{M}_K$ through which all other cohomology functors factor. This means there are **realization functors** like $\mathrm{Real}_p : \mathcal{M}_K \to$ (p-adic representations of G_K), $\mathrm{Real}_\iota : \mathcal{M}_K \to \mathrm{Ab}$, and $\mathrm{Real}_{dR} : \mathcal{M}_K \to$(filtered K-vector spaces) such that $H^i_{\text{ét}}(X, \mathbb{Q}_p) = \mathrm{Real}_p \circ H^i(X)$, etc. There should also be other properties: comparison for various K, existence of tensor products and dual objects in $\mathcal{M}_K$, etc.

Conjecture 6 (Grothendieck-Serre)**.** *$H^i(X, \mathbb{Q}_p)$ is a semi-simple representation of G_K.*

This is known for abelian varieties and in a few other cases.

Conjecture 7 (Tate)**.** *$(H^{2q}(X, \mathbb{Q}_p)(q))^{G_K}$ is generated over $\mathbb{Q}_p$ by the classes of sub-varieties Z of codimension q.*

These two conjectures imply that the functor Real_p is fully faithful, hence the category $\mathcal{M}_K$ is equivalent to the category of p-adic representations of G_K coming from geometry (i.e. subquotients of $H^i(X, \mathbb{Q}_p)(n)$ for some i, n, and X).

Let W be the extension $1 \to V \to W \to \mathbb{Q}_p \to 1$ given by some nonzero $x \in H^1_g(V, G_K)$. We cannot expect this representation to come from a pure motive, but we do expect it comes from a **mixed motive**. The category $\mathcal{MM}_K$ of mixed motives should be not semi-simple or graded in any interesting way, and it should contain $\mathcal{M}_K$ as a full subcategory. It should be to $\mathcal{M}_K$ as (varieties over K) is to VPS_K. It is expected that Real_p induces an isomorphism

$$\mathrm{Ext}^1_{\mathcal{MM}_K}(Q, M) \cong H^1_g(G_K, V)$$

where $\mathrm{Real}_p(Q) = \mathbb{Q}_p$ and $\mathrm{Real}_p(M) = V$. It is possible to define what should be $\mathrm{Ext}^1_{\mathcal{MM}_K}(Q, M)$ when $M = H^i(X)$ using the K-theory of X, and this case is the other conjecture of Bloch and Kato. It has been proven by Voevodsky.

Here are a few remarks on the literature. See [Tat95] for more information on the Birch and Swinnerton-Dyer conjecture. For background information necessary to understand the conjecture, see [FO]. See [BK90] for Bloch and Kato's conjecture and [FPR94] for its extension by Fontaine and Perrin-Riou.

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