Intriguing Problems for Students in a Proofs Class

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Outline

Induction: The Square

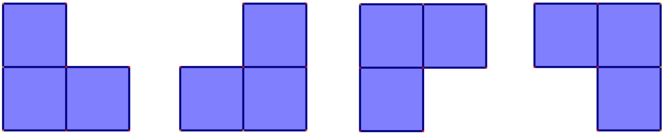
Prove that you can cut a square into n squares for any $n \geq 6$.

Induction: The Chocolate

If a chocolate bar is made of n 1×1 pieces, how many times do you need to break it to separate it into 1×1 pieces? (You can only break one piece at a time).

Induction: L-omino tiling

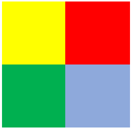
Explain how to tile a $2^n \times 2^n$ checkerboard with one square missing using L-ominoes:

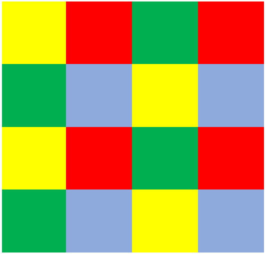


See the Kadon Enterprises, Inc. booth at the JMM exhibits to see this in action!

Induction: 2×2-Coloring

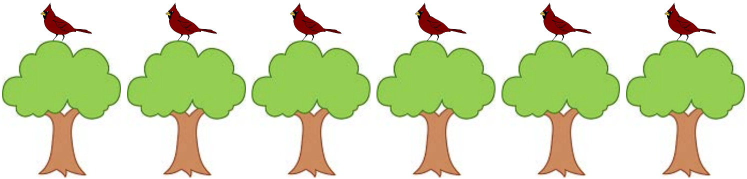
Let's say a chessboard is 2×2-colored if it is colored in 4 colors such that every 2×2 area is colored in all 4 distinct colors. Prove that the four corners of a 2×2-colored 1000×1000 board are colored in all 4 distinct colors.





Numerical Invariant: The Birds

There are 6 trees in a row, one bird on each. Each hour two birds take off and each lands on a tree adjacent to where it was. Can they ever all end up on the same tree? (Hint: parity)



Pigeonhole Principle: Divisibility

Show that in any set of n numbers, there is a subset whose sum is divisible by n.

Pigeonhole Principle: Independence Problems

How many knights/queens/kings/bishops can you put on a chessboard so no piece can hit another in one move?

e.g. Kings:

K		K		K		K	
K		K		K		K	
K		K		K		K	
K		K		K		K	

Sources

- *Algebra and Number Theory for Mathematical Schools* by Alfutova, N.B. and Oostinov, A.V. (in Russian)
- *Everything You Always Wanted To Know About Mathematics* (*But didn't even know to ask): A Guided Journey Into the World of Abstract Mathematics and the Writing of Proofs* by Brendan Sullivan with John Mackey
- A Math Teachers' Circle problem related to me by Maksym Fedorchuk

Thank you so much
for your attention!

Multiplying Pieces

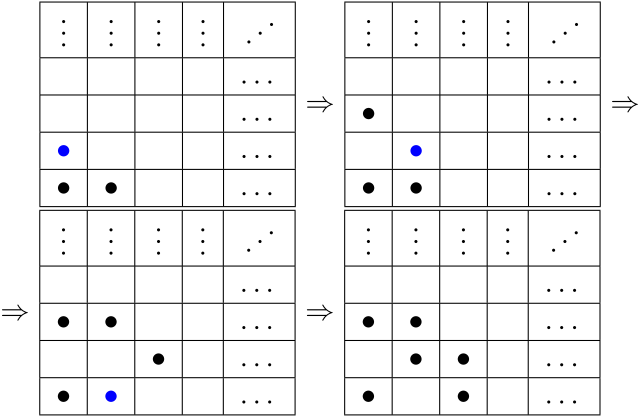
You start with three pieces (say circles) in the lower-left corner of a grid that extends infinitely far to the right and up:

$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
					$\dots$
					$\dots$
					$\dots$
●					$\dots$
●	●				$\dots$

A valid move is to replace a piece by two pieces: one to the right of the original and one just above the original, provided both of those spots are empty.

Multiplying Pieces

Example sequence of valid moves:



Question. Can you ever move all the pieces out of the three bottom-left spaces?

Multiplying Pieces: Hint

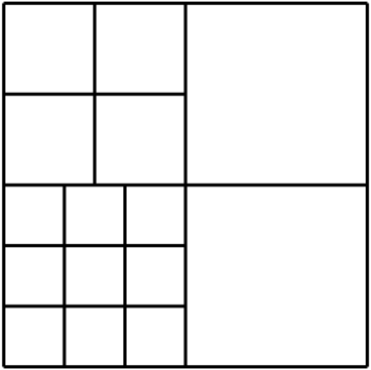
$\iddots$	$\iddots$	$\iddots$	$\iddots$	$\iddots$	$\iddots$
2^{-4}	$\iddots$	$\iddots$	$\iddots$	$\iddots$	$\iddots$
2^{-3}	2^{-4}	$\iddots$	$\iddots$	$\iddots$	$\iddots$
2^{-2}	2^{-3}	2^{-4}	$\iddots$	$\iddots$	$\iddots$
2^{-1}	2^{-2}	2^{-3}	2^{-4}	$\iddots$	$\iddots$
1	2^{-1}	2^{-2}	2^{-3}	2^{-4}	$\iddots$

The sum of these numbers over all places where there are pieces is invariant.

The Square: Hint

Prove that you can cut a square into n squares for any $n \geq 6$.

$n = 15$



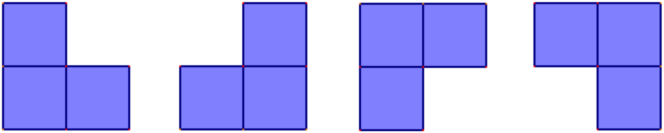
The Chocolate Bar: Hint

If a chocolate bar is made of n 1×1 pieces, how many times do you need to break it to separate it into 1×1 pieces?

Hint: Break it anywhere! Count how many squares remain in each piece.

L-omino tiling: Hint

Explain how to tile a $2^n \times 2^n$ checkerboard with one square missing using L-ominoes:



Hint: Break up the board into quarters.

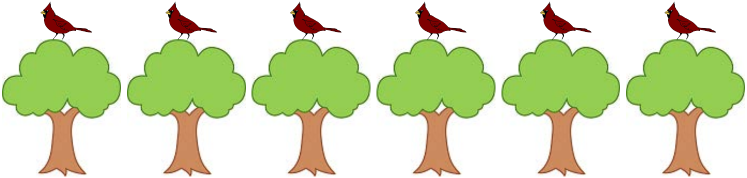
2×2-Coloring: Hint

Let's say a chessboard is 2×2-colored if it is colored in 4 colors such that every 2×2 area is colored in all 4 distinct colors. Prove that the four corners of a 2×2-colored 1000×1000 board are colored in all 4 distinct colors.

Hint: Prove the $2 \times 2N$ case, then use that and induction on M to prove the $2M \times 2N$ case. Show that the two left corners have colors disjoint from the two right corners.

The Birds: Hint

Six trees, one bird on each. Each hour two birds take off and each lands on a tree adjacent to where it was. Can they ever all end up on the same tree?



Hint: Consider the sum of distances from some tree $\pmod 2$, or the sum of birds on every other tree $\pmod 2$

Divisibility: Hint

Show that in any set of n numbers, there is a subset whose sum is divisible by n.

Hint:

$$
\begin{aligned}
&x_1,\\
&x_1 + x_2,\\
&x_1 + x_2 + x_3,\\
&\vdots\\
&x_1 + x_2 + x_3 + \cdots + x_n.
\end{aligned}
$$