Weil-étale Cohomology

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Abstract

We will be talking about a subject, almost no part of which is yet completely defined. I will introduce the Weil group, Grothendieck topologies (if needed), and the Weil-etale topology which Steve Lichtenbaum defined for schemes over a finite field. I will also talk about how it relates to special values of zeta-functions.

1 Introduction

Today we will be talking about Professor Lichtenbaum’s paper on the Weil-étale topology and what turned out to be a better definition given by Thomas Geisser.

2 Definitions and Basics

$X$ = scheme of finite type over $k = \mathbb{F}_q$.
$\breve{X} = X \times_k \bar{k}$.

Definition 1. Let $G$ be a group of automorphisms of a scheme $X$. We say $G$ acts on a sheaf $F$ on the étale topology on $X$ if we have a compatible system of maps $\psi_\sigma : F \to \sigma_* F$ for all $\sigma \in G$ [i.e. $\psi_{\sigma \tau} = \sigma_* \psi_\tau \circ \psi_\sigma$ and $\psi_{id} = id$].

Note that we have a Galois action of $\hat{\mathbb{Z}} = \text{Gal}(\bar{k}/k)$ on $\breve{X}$ over $X$. Let $\Gamma_0 \cong \mathbb{Z}$ denote the powers of Frobenius in $\hat{\mathbb{Z}}$.

Definition 2. A Weil-étale sheaf $F$ on $X$ is an étale sheaf on $\breve{X}$ with a $\Gamma_0$-action.
Definition 3. The Weil-étale cohomology is defined by \( H^0_W(X,F) = \Gamma_W(F) = F(\bar{X})^\mathbb{Z} \), and \( H^i_W(X,F) \) are defined to be the right derived functors of \( H^0_W \).

Here are the tools that are used to compute Weil-étale cohomology:

Let \( \rho : \text{Sh}_{W,X} \to \text{Sh}_{\acute{e}t,\bar{X}} \) be the forgetful functor, and \( \phi : \text{Sh}_{\acute{e}t,X} \to \text{Sh}_{W,X} \) take \( F \) to \( \pi_1^*F \) with the induced action of \( \Gamma_0 \).

Fact 1. For \( F \in \text{Sh}_{\acute{e}t,X} \) there is a functorial map of spectral sequences from
\[
H^p(\hat{Z}, H^q_{\acute{e}t}(\bar{X}, \pi_1^*F)) \Rightarrow H^{p+q}_{\acute{e}t}(X,F)
\]
to
\[
H^p(Z, H^q_{\acute{e}t}(\bar{X}, \pi_1^*(F))) \Rightarrow H^{p+q}_W(X,\phi(F))
\]
Since \( \text{cd} \mathbb{Z} = 1 \), the last spectral sequence breaks up into short exact sequences
\[
0 \to H^1(Z, H^q_{\acute{e}t}(\bar{X}, \pi_1^*(F))) \to H^{q+1}_{\acute{e}t}(X,\phi(F)) \to H^0(Z, H^{q+1}_{\acute{e}t}(\bar{X}, \pi_1^*(F))) \to 0
\]
Finally, there are natural maps \( c_i : H^i_{\acute{e}t}(X,F) \to H^i_W(X,\phi(F)) \) which are isomorphisms when the sheaf \( F \) is torsion.

3 Duality Theorem for Non-singular Curves

We define the sheaf \( \mathbb{G}_m \) on \( W_X \) by \( \phi(\mathbb{G}_{m,X}) \), where \( \mathbb{G}_{m,X} \) is the sheaf \( \mathbb{G}_m \) on the étale topology on \( X \).

Theorem 1. Suppose \( U \) is a smooth curve over \( k \), \( \bar{U} \) connected. Then \( H^q_W(U, \mathbb{G}_m) \) is finitely generated for all \( q \) and zero for \( q \geq 3 \). If \( U \) is also projective then
\[
H^q_W(U, \mathbb{G}_m) = \begin{cases}
\mathbb{K}^*, & q = 0 \\
\text{Pic}(U), & q = 1 \\
\mathbb{Z}, & q = 2 \\
0, & q \geq 3
\end{cases}
\]

The étale cohomology of \( \mathbb{G}_m \) is the same for \( q = 0,1; 0 \) for \( q = 2 \); and \( \mathbb{Q}/\mathbb{Z} \) for \( q = 3 \). [Milne’s notes on étale cohomology] In the situation of theorem 1, let \( j : U \to X \) be an open dense embedding of \( U \) in a smooth projective curve \( X \) over \( k \), and let \( F \in \text{Sh}_{W,X} \). Since \( X \) has finite cohomological dimension \( [\text{cd} X < \infty \) in the étale topology, and the second spectral
Theorem 2.

The significance of this theorem is that \( \kappa_{j;Z} \) is the duality isomorphism showing that the cohomology of \( \mathbb{G}_m \) is dual, with respect to \( \text{RHom}(-, \mathbb{Z}) = \text{RHom}(-, \mathbb{Z}[-2]) \), to the cohomology of \( \mathbb{Z} \).

**Proof.** First let \( F = j_!(\mathbb{Z}/n\mathbb{Z}) \). [I will mean \( F \) as both an étale and Weil-étale sheaf] The following diagram is commutative in \( D(Ab) \):

\[
\begin{align*}
\text{RHom}_{\text{ét},X}(F, \mathbb{G}_m) & \xrightarrow{\text{RΓ}_\text{ét}} \text{RHom}_{Ab}(\text{RΓ}_\text{ét}F, \text{RΓ}_\text{ét}(\mathbb{G}_m)) \xrightarrow{\alpha} \text{RHom}_{Ab}(\text{RΓ}_\text{ét, X}F, \mathbb{Q}/\mathbb{Z}[-3]) \xrightarrow{\phi} \text{RHom}_{Ab}(\text{RΓ}_\text{ét}F, \mathbb{Z}[-2]) \\
\text{RHom}_W(F, \mathbb{G}_m) & \xrightarrow{\text{RΓ}_W} \text{RHom}_{Ab}(\text{RΓ}_W F, \text{RΓ}_W(\mathbb{G}_m)) \xrightarrow{\beta} \text{RHom}_{Ab}(\text{RΓ}_W F, \mathbb{Z}[-2])
\end{align*}
\]

Deninger \( \Rightarrow \alpha \circ \text{RΓ}_\text{ét} \) is the duality isomorphism interpreting class field theory for function fields in terms of étale cohomology. [The homology of \( \text{RHom}_{\text{ét}}(F, \mathbb{G}_m) \) is \( \text{Ext}^1(X, F, \mathbb{G}_m) \) and that of \( \text{RHom}_{D(Ab)}(\text{RΓ}_\text{ét}F, \mathbb{Q}/\mathbb{Z}[-3]) \) is the Pontryagin dual of \( H^{3-i}(X, F) \). The induced maps on homology come from the Yoneda pairing, and Deninger shows they are isomorphisms, hence the maps induce a quasi-isomorphism, so we get an isomorphism in the derived category.]

The rightmost map is an isomorphism because \( \phi : \text{RΓ}_\text{ét}F \sim \text{RΓ}_W F \) and \( \text{RHom}_{D(Ab)}(\text{RΓ}_\text{ét}F, \mathbb{Q}) = 0 \) [since \( \text{RΓ}_\text{ét}F \) is killed by \( n \)] so the exact sequence \( 0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Q} \rightarrow \mathbb{Q}/\mathbb{Z} \rightarrow 0 \) induces the exact sequence \( \text{RHom}(\text{RΓ}_\text{ét}F, \mathbb{Q}[-3]) \rightarrow \text{RHom}(\text{RΓ}_\text{ét}F, \mathbb{Q}/\mathbb{Z}[-3]) \rightarrow \text{RHom}(\text{RΓ}_\text{ét}F, \mathbb{Z}[-2]) \rightarrow \text{RHom}(\text{RΓ}_\text{ét}F, \mathbb{Q}[-2]) \) where the first and last terms are 0. It is now enough to show that the leftmost map is an isomorphism.

We prove that \( \phi \) takes the étale version of different sheaves to the same sheaves in the Weil-étale topology, i.e. preserves them. Let \( Z = X \setminus U \) with the induced reduced structure and \( i : Z \rightarrow X \) the natural inclusion. First
you show that $\phi$ preserves $\text{Ext}^q(i_*\mathbb{Z}, \mathbb{G}_m)$. [Details: in both the étale and Weil-étale topologies, there is a natural isomorphism $\text{Hom}(i_*\mathbb{Z}, F) \cong i^!F$ for any sheaf $F$, so $\text{Ext}^q(i_*\mathbb{Z}, F) \cong R^q i^! F$, and $R^q i^! \mathbb{G}_m = 0$ for $q \neq 1$ and $\mathbb{Z}$ if $q = 1$.] Since $\text{Ext}^q(\mathbb{Z}, \mathbb{G}_m) = \mathbb{G}_m$ if $q = 0$ and $0$ otherwise, $\phi$ preserves $\text{Ext}^q(\mathbb{Z}, \mathbb{G}_m)$. The exact sequence

$$0 \rightarrow j_! \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow i_* \mathbb{Z} \rightarrow 0$$

then shows that $\phi$ preserves $\text{Ext}^q(j_! \mathbb{Z}, \mathbb{G}_m)$. Then the exact sequence

$$0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}/n \mathbb{Z} \rightarrow 0$$

shows $\phi$ preserves $\text{Ext}^q(j_!(\mathbb{Z}/n \mathbb{Z}), \mathbb{G}_m)$. Finally, the map of local to global spectral sequences

$$H^p_{\text{ét}}(X, \text{Ext}^q_{\text{ét}}(j_!(\mathbb{Z}/n \mathbb{Z}), \mathbb{G}_m)) \Rightarrow \text{Ext}^{p+q}_{\text{ét}}(j_!(\mathbb{Z}/n \mathbb{Z}), \mathbb{G}_m)$$

to

$$H^p_W(X, \text{Ext}^q_W(j_!(\mathbb{Z}/n \mathbb{Z}), \mathbb{G}_m)) \Rightarrow \text{Ext}^{p+q}_W(j_!(\mathbb{Z}/n \mathbb{Z}), \mathbb{G}_m)$$

which is an isomorphism on the $E_2^{pq}$ terms since the sheaves are torsion, hence on the $E^n$ terms, shows that the left map induced by $\phi$ is a quasi-isomorphism, hence an isomorphism in the derived category. Thus the proof for $F = j_!(\mathbb{Z}/n \mathbb{Z})$ is done.

Now we have to prove this for $F = j_!\mathbb{Z}$. We have the following map of triangles in $D(\mathbb{Z})$:

$$
\begin{array}{cccc}
M & \xrightarrow{\alpha} & M & \rightarrow & M_n & \rightarrow & M[1] \\
\downarrow g & & \downarrow g & & \downarrow g_n & & \downarrow g \\
N & \xrightarrow{\beta} & N & \rightarrow & N_n & \rightarrow & N[1]
\end{array}
$$

where $M' = R\text{Hom}_{W,X}(j_!\mathbb{Z}, \mathbb{G}_m)$, $N' = R\text{Hom}_{W}(R\Gamma_{V,X}j_!\mathbb{Z}, \mathbb{Z}[-2])$, $M_n$ and $N_n$ are the same with $j_!\mathbb{Z}$ replaced by $j_!(\mathbb{Z}/n \mathbb{Z})$, $g = \kappa_{j_!\mathbb{Z}}$, and $g_n = \kappa_{j_!(\mathbb{Z}/n \mathbb{Z})}$ is a quasi-isomorphism. It is an easy fact from derived categories that, in this scenario, if the homology groups of $M'$ and $N'$ are finitely generated then $g$ is a quasi-isomorphism.

And indeed they are, because $R\text{Hom}_{W,X}(j_!\mathbb{Z}, (\mathbb{G}_m)_X) \cong R\text{Hom}_{W,U}(\mathbb{Z}, (\mathbb{G}_m)_U) \cong R\Gamma_{W,U}(\mathbb{G}_m)$, whose cohomology groups are finitely generated and zero for
large $i$ by theorem 1. [Details: $j^*$ has the exact left adjoint $j_!$, so $j^*$ takes injectives to injectives. And $j^*$ is exact, so it carries a resolution of $\mathbb{G}_m_X$ to a resolution of $j^*(\mathbb{G}_m)_X = (\mathbb{G}_m)_U$. $\text{Hom}_U(\mathbb{Z}, (\mathbb{G}_m)_U) \cong \Gamma_U(\mathbb{G}_m)$ - same proof as usual in algebraic geometry - implies $\text{RHom}_U(\mathbb{Z}, (\mathbb{G}_m)_U) \cong \Gamma_U(\mathbb{G}_m)$.]

\section{Zeta Functions}

Let $\Gamma_0$ denote the subgroup $\mathbb{Z} \subseteq \hat{\mathbb{Z}}$ generated by the Frobenius automorphism. Then $H^1(k, \mathbb{Z}) = H^1(\Gamma_0, \mathbb{Z}) = \text{Hom}(\Gamma_0, \mathbb{Z})$ [The category of étale sheaves on $\overline{k}$ is $\text{Ab}$; the category of Weil-étale sheaves is $\mathbb{Z}[\mathbb{Z}]$-modules]. Let $\theta \in H^1(k, \mathbb{Z})$ represent the element corresponding to the homomorphism taking $\text{Frob}$ to 1. For any scheme $X$ over $k$, we also denote by $\theta \in H^1(\overline{k}, \mathbb{Z})$ the pullback of $\theta \in H^1(k, \mathbb{Z})$. For any Weil-étale sheaf $F$ there is a natural pairing $F \otimes \mathbb{Z} \to F$ induced by $x \otimes n \mapsto nx$. This pairing in turn induces a map $\cup \theta : H^i_W(X, F) \to H^{i+1}_W(X, F)$. $\theta \cup \theta = 0$ [since it is induced from the pullback of $\theta \cup \theta \in H^2(k, \mathbb{Z}) = 0$] so this makes the cohomology groups $(H^i_W(X, F))$ into a complex. Let $h^i_W(X, F)$ be the homology groups of this complex.

Conjecture 1. $U$ is a quasi-projective variety over $k$, $j : U \to X$ an open dense immersion of $U$ in a projective variety $X$. If $U$ is smooth or $U$ is a curve then:

(1) The cohomology groups $H^i_W(X, j_! \mathbb{Z})$ are finitely generated abelian groups which are zero for large $i$ and are independent of $j$ and $X$.

(2) Let $r_i$ be the rank of $H^i_W(X, j_! \mathbb{Z})$. Then $\sum (-1)^i r_i = 0$.

(3) The order $a_U$ of the zero of $Z(U, t)$ at $t = 1$ is $\sum (-1)^i r_i$.

(4) The homology groups $h^i_W(X, j_! \mathbb{Z})$ are finite.

(5) $\lim_{t \to 1} Z(U, t)(1 - t)^{-a_U} = \pm \chi(X, j_! \mathbb{Z}) = \prod_i |h^i(X, j_! \mathbb{Z})|^{(-1)^i}$.

where $Z(U, t) = \exp(\sum_{r=1}^\infty N_r t^r / r)$.

Theorem 3. The conjecture is true if $U$ is projective and smooth, or if $U$ is a smooth surface with $X$ smooth, or if $U$ is any curve.
5  Geisser’s Definition

Unfortunately, virtually every part of Professor Lichtenbaum’s conjecture that was not proven is false, as shown by Geisser and Weibel. So, following the idea of Voevodsky of defining a Grothendieck topology generated by Nisnevich covers and abstract blowups, Geisser made the following definition.

**Definition 4.** The étale h-topology, or eh-topology on a subcategory of the category of schemes is the Grothendieck topology generated by étale coverings and abstract blowups, meaning if we have a cartesian square

\[
\begin{array}{ccc}
Z' = Z \times_X X' & \rightarrow & X' \\
\downarrow & & \downarrow \\
Z & \rightarrow & X
\end{array}
\]

where \( f \) is proper, \( i \) is a closed embedding, and \( f \) induces an isomorphism \( X' \setminus Z' \iso X \setminus Z \), then \( \{ X' \xrightarrow{f} X, Z \xrightarrow{i} X \} \) is a covering.

Voevodsky’s topology has better properties for singular schemes than Nisnevich’s topology, and likewise the eh-topology has better properties than the étale topology. Following are some examples of covers in this topology:

1. A scheme \( X \) is covered by its irreducible components.
2. \( X^{\text{red}} \rightarrow X \) is a covering.
3. For a blowup \( X' \) of \( X \) with center \( Z \), \( \{ X' \rightarrow X, Z \rightarrow X \} \) is a covering.
4. A proper morphism \( p : X' \rightarrow X \) such that \( \forall x \in X \exists y \in p^{-1}(x) | k(y) \cong k(x) \) [they have the same residue field] is a covering, called a proper eh-covering.

**Proposition 1.** Every eh-covering of \( X \) has a refinement of the form \( \{ U_i \rightarrow X' \rightarrow X \}_{i \in I} \) where \( \{ U_i \rightarrow X \}_{i \in I} \) is an étale cover and \( X' \rightarrow X \) is a proper eh-covering.

**Definition 5.** A Weil eh-sheaf \( F \) on \( \text{Sch}/\overline{\mathbb{F}}_q \) is an eh-sheaf on \( \text{Sch}/\overline{\mathbb{F}}_q \) together with a \( \mathbb{Z} \)-action. Weil-eh cohomology is defined as before by the derived functors of \( F \mapsto F(\bar{X})^{\mathbb{Z}} \).

With this definition, all the corresponding parts of Professor Lichtenbaum’s conjecture are true.
6 Details

The following commutative diagram of functors induces the map of spectral sequences:

\[
\begin{array}{ccc}
\tilde{X}_{\text{ét}} & \rightarrow & \tilde{k}_{\text{ét}} \\
\downarrow^{\phi} & & \downarrow \\
\text{Sh}_{W,X} & \rightarrow & \mathbb{Z}\text{-mod} \\
\end{array}
\]

Theorem 4. If \(X\) is connected, then

\[
H^q_{W}(X,\mathbb{Z}) = \begin{cases} 
\mathbb{Z}, & q = 0, 1 \\
H^2_{\text{ét}}(X,\mathbb{Z})/\mathbb{Q}/\mathbb{Z}, & q = 2 \\
H^q_{\text{ét}}(X,\mathbb{Z}), & q \geq 3 
\end{cases}
\]

Otherwise, \(X\) is a finite disjoint union of schemes. In either case, all these groups are finite.

The étale cohomology is finite, 0 for \(q = 1\) and \(q \gg 0\), \(\mathbb{Q}/\mathbb{Z}\oplus\) (finite group) for \(q = 2\), and \(\mathbb{Z}\) for \(q = 0\). [Milne’s paper, Values of Zeta Functions of Varieties over Finite Fields]

Theorem 5. \(U\) = smooth \(d\)-dimensional quasi-projective variety over \(k, d \leq 2\). Let \(j: U \rightarrow X\) be the open immersion into a resolution of singularities \(X\) for \(U\) (i.e. \(X\) is smooth projective). Then \(H^q_{W}(X,j_!\mathbb{Z})\) is finitely generated for all \(q\), zero for \(q\) large, and independent of the choices of \(j\) and \(X\).

Key idea used twice in the proof: suppose \(j: U \rightarrow X\) and \(j': U \rightarrow X'\) are two such immersions. Replace \(X'\) by the closure of the image of \(U\) in \(X \times X'\) induced by \(j\) and \(j'\) (or by the smooth projectivization of the latter). Then there is a map \(\pi: X' \rightarrow X\) such that \(\pi \circ j' = j\), and it turns out \(\pi_*j'_!\mathbb{Z} = j_!\mathbb{Z}\) and you go from there.