Weil-étale Cohomology

Igor Minevich

March 13, 2012

Abstract

We will be talking about a subject, almost no part of which is yet completely defined. I will introduce the Weil group, Grothendieck topologies (if needed), and the Weil-etale topology which Steve Lichtenbaum defined for schemes over a finite field. I will also talk about how it relates to special values of zeta-functions.

1 Introduction

Today we will be talking about Professor Lichtenbaum's paper on the Weil-étale topology and what turned out to be a better definition given by Thomas Geisser.

2 Definitions and Basics

$X =$ scheme of finite type over $k = \mathbb{F}_q$.
$\bar{X} = X \times_k \bar{k}$.

Definition 1. *Let G be a group of automorphisms of a scheme X. We say G acts on a sheaf F on the étale topology on X if we have a compatible system of maps $\psi_\sigma : F \to \sigma_* F$ for all $\sigma \in G$ [i.e. $\psi_{\sigma\tau} = \sigma_* \psi_\tau \circ \psi_\sigma$ and $\psi_{id} = id$].*

Note that we have a Galois action of $\hat{Z} = \mathrm{Gal}(\bar{k}/k)$ on $\bar{X}$ over X. Let $\Gamma_0 \cong \mathbb{Z}$ denote the powers of Frobenius in $\hat{Z}$.

Definition 2. *A Weil-étale sheaf F on X is an étale sheaf on $\bar{X}$ with a Γ_0-action.*

Definition 3. *The Weil-étale cohomology is defined by* $H^0_{\mathcal{W}}(X,F) = \Gamma_{\mathcal{W}}(F) = F(\bar{X})^{\mathbb{Z}}$*, and* $H^i_{\mathcal{W}}(X,F)$ *are defined to be the right derived functors of* $H^0_{\mathcal{W}}$*.*

Here are the tools that are used to compute Weil-étale cohomology:
Let $\rho : \mathrm{Sh}_{\mathcal{W},X} \to \mathrm{Sh}_{\text{ét},\bar{X}}$ be the forgetful functor, and $\phi : \mathrm{Sh}_{\text{ét},X} \to \mathrm{Sh}_{\mathcal{W},X}$ take F to $\pi_1^* F$ with the induced action of Γ_0.

Fact 1. *For* $F \in \mathrm{Sh}_{\text{ét},X}$ *there is a functorial map of spectral sequences from*

$$H^p(\hat{\mathbb{Z}}, H^q_{\text{ét}}(\bar{X}, \pi_1^* F)) \Rightarrow H^{p+q}_{\text{ét}}(X,F)$$

to

$$H^p(\mathbb{Z}, H^q_{\text{ét}}(\bar{X}, \pi_1^*(F))) \Rightarrow H^{p+q}_{\mathcal{W}}(X, \phi(F))$$

Since $\operatorname{cd}\mathbb{Z} = 1$*, the last spectral sequence breaks up into short exact sequences*

$$0 \to H^1(\mathbb{Z}, H^q_{\text{ét}}(\bar{X}, \pi_1^*(F))) \to H^{q+1}_{\mathcal{W}}(X, \phi(F)) \to H^0(\mathbb{Z}, H^{q+1}_{\text{ét}}(\bar{X}, \pi_1^*(F))) \to 0$$

Finally, there are natural maps $c_i : H^i_{\text{ét}}(X,F) \to H^i_{\mathcal{W}}(X,\phi(F))$ *which are isomorphisms when the sheaf* F *is torsion.*

3 Duality Theorem for Non-singular Curves

We define the sheaf $\mathbb{G}_m$ on $\mathcal{W}_X$ by $\phi(\mathbb{G}_{m,X})$, where $\mathbb{G}_{m,X}$ is the sheaf $\mathbb{G}_m$ on the étale topology on X.

Theorem 1. *Suppose* U *is a smooth curve over* k*,* $\bar{U}$ *connected. Then* $H^q_{\mathcal{W}}(U, \mathbb{G}_m)$ *is finitely generated for all* q *and zero for* $q \geq 3$*. If* U *is also projective then*

$$H^q_{\mathcal{W}}(U,\mathbb{G}_m) = \begin{cases} k^*, & q = 0 \\ \mathrm{Pic}(U), & q = 1 \\ \mathbb{Z}, & q = 2 \\ 0, & q \geq 3 \end{cases}$$

The étale cohomology of $\mathbb{G}_m$ is the same for $q = 0, 1$; 0 for $q = 2$; and $\mathbb{Q}/\mathbb{Z}$ for $q = 3$. [Milne's notes on étale cohomology] In the situation of theorem 1, let $j : U \to X$ be an open dense embedding of U in a smooth projective curve X over k, and let $F \in \mathrm{Sh}_{\mathcal{W},X}$. Since X has finite cohomological dimension [$\operatorname{cd} X < \infty$ in the étale topology, and the second spectral sequence then shows $\operatorname{cd} X < \infty$ for the Weil-étale topology], we can define $R\Gamma_{\mathcal{W}} F \in D^b(\mathrm{Sh}_{\mathcal{W},X})$ [by taking an injective resolution of F, and replacing F by a sufficiently far-out truncation]. By applying Γ_X we get a natural map in $D(Ab)$ from $\mathrm{RHom}_{\mathcal{W}}(F, \mathbb{G}_m)$ to $\mathrm{RHom}_{D(Ab)}(R\Gamma_{\mathcal{W}}(F), R\Gamma_{\mathcal{W}}(\mathbb{G}_m))$. Then, since $H^2_{\mathcal{W}}(X, \mathbb{G}_m) = \mathbb{Z}$ and $H^q_{\mathcal{W}}(X, \mathbb{G}_m) = 0$ for $q \geq 3$ we get a natural map in $D(Ab)$ from $R\Gamma_{\mathcal{W}}(\mathbb{G}_m)$ to $\mathbb{Z}[-2]$. Composing with the previous map, we get a map $\kappa_F : \mathrm{RHom}_{\mathcal{W}}(F, \mathbb{G}_m) \to \mathrm{RHom}_{D(Ab)}(R\Gamma_{\mathcal{W}}(F), \mathbb{Z}[-2])$.

Theorem 2. *κ_F is an isomorphism when $F = j_!\mathbb{Z}$ or $j_!(\mathbb{Z}/n\mathbb{Z})$.*

The significance of this theorem is that $\kappa_{j_!\mathbb{Z}}$ is the duality isomorphism showing that the cohomology of $\mathbb{G}_m$ is dual, with respect to $\mathrm{RHom}(-, \mathbb{Z}) = \mathrm{RHom}(-, \mathbb{Z}[-2])$, to the cohomology of $\mathbb{Z}$.

Proof. First let $F = j_!(\mathbb{Z}/n\mathbb{Z})$. [I will mean F as both an étale and Weil-étale sheaf] The following diagram is commutative in $D(Ab)$:

$$\begin{array}{ccccc}
\mathrm{RHom}_{\text{ét},X}(F, \mathbb{G}_m) & \xrightarrow{R\Gamma_{\text{ét}}} & \mathrm{RHom}_{Ab}(R\Gamma_{\text{ét}} F, R\Gamma_{\text{ét}}(\mathbb{G}_m)) & \xrightarrow{\alpha} & \mathrm{RHom}_{Ab}(R\Gamma_{\text{ét},X} F, \mathbb{Q}/\mathbb{Z}[-3]) \\
\downarrow^{\phi} & & \downarrow^{\phi} & & \downarrow^{\phi} \\
\mathrm{RHom}_{\mathcal{W}}(F, \mathbb{G}_m) & \xrightarrow{R\Gamma_{\mathcal{W}}} & \mathrm{RHom}_{Ab}(R\Gamma_{\mathcal{W}} F, R\Gamma_{\mathcal{W}}(\mathbb{G}_m)) & \xrightarrow{\beta} & \mathrm{RHom}_{Ab}(R\Gamma_{\mathcal{W}} F, \mathbb{Z}[-2])
\end{array}$$

Deninger $\Rightarrow \alpha \circ R\Gamma_{\text{ét}}$ is the duality isomorphism interpreting class field theory for function fields in terms of étale cohomology. [The homology of $\mathrm{RHom}_{\text{ét}}(F, \mathbb{G}_m)$ is $\mathrm{Ext}^i_X(F, \mathbb{G}_m)$ and that of $\mathrm{RHom}_{D(Ab)}(R\Gamma_{\text{ét}} F, \mathbb{Q}/\mathbb{Z}[-3])$ is the Pontryagin dual of $H^{3-i}(X, F)$. The induced maps on homology come from the Yoneda pairing, and Deninger shows they are isomorphisms, hence the maps induce a quasi-isomorphism, so we get an isomorphism in the derived category.]

The rightmost map is an isomorphism because $\phi : R\Gamma_{\text{ét}} F \xrightarrow{\sim} R\Gamma_{\mathcal{W}} F$ and $\mathrm{RHom}_{D(Ab)}(R\Gamma_{\text{ét}} F, \mathbb{Q}) = 0$ [since $R\Gamma_{\text{ét}} F$ is killed by n] so the exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ induces the exact sequence $\mathrm{RHom}(R\Gamma_{\text{ét}} F, \mathbb{Q}[-3]) \to \mathrm{RHom}(R\Gamma_{\text{ét}} F, \mathbb{Q}/\mathbb{Z}[-3]) \to \mathrm{RHom}(R\Gamma_{\text{ét}} F, \mathbb{Z}[-2]) \to \mathrm{RHom}(R\Gamma_{\text{ét}} F, \mathbb{Q}[-2])$ where the first and last terms are 0. It is now enough to show that the leftmost map is an isomorphism.

We prove that ϕ takes the étale version of different sheaves to the same sheaves in the Weil-étale topology, i.e. preserves them. Let $Z = X \setminus U$ with the induced reduced structure and $i : Z \to X$ the natural inclusion. First you show that ϕ preserves $\underline{\mathrm{Ext}}^q(i_*\mathbb{Z}, \mathbb{G}_m)$. [Details: in both the étale and Weil-étale topologies, there is a natural isomorphism $\underline{\mathrm{Hom}}(i_*\mathbb{Z}, F) \cong i^! F$ for any sheaf F, so $\underline{\mathrm{Ext}}^q(i_*\mathbb{Z}, F) \cong R^q i^! F$, and $R^q i^! \mathbb{G}_m = 0$ for $q \neq 1$ and $\mathbb{Z}$ if $q = 1$.] Since $\underline{\mathrm{Ext}}^q(\mathbb{Z}, \mathbb{G}_m) = \mathbb{G}_m$ if $q = 0$ and 0 otherwise, ϕ preserves $\underline{\mathrm{Ext}}^q(\mathbb{Z}, \mathbb{G}_m)$. The exact sequence

$$0 \to j_! \mathbb{Z} \to \mathbb{Z} \to i_* \mathbb{Z} \to 0$$

then shows that ϕ preserves $\underline{\mathrm{Ext}}^q(j_!\mathbb{Z}, \mathbb{G}_m)$. Then the exact sequence

$$0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$$

shows ϕ preserves $\underline{\mathrm{Ext}}^q(j_!(\mathbb{Z}/n\mathbb{Z}), \mathbb{G}_m)$. Finally, the map of local to global spectral sequences

$$H^p_{\text{ét}}(X, \underline{\mathrm{Ext}}^q_{\text{ét}}(j_!(\mathbb{Z}/n\mathbb{Z}), \mathbb{G}_m)) \Rightarrow \mathrm{Ext}^{p+q}_{\text{ét}}(j_!(\mathbb{Z}/n\mathbb{Z}), \mathbb{G}_m)$$

to

$$H^p_{\mathcal{W}}(X, \underline{\mathrm{Ext}}^q_{\mathcal{W}}(j_!(\mathbb{Z}/n\mathbb{Z}), \mathbb{G}_m)) \Rightarrow \mathrm{Ext}^{p+q}_{\mathcal{W}}(j_!(\mathbb{Z}/n\mathbb{Z}), \mathbb{G}_m)$$

which is an isomorphism on the E_2^{pq} terms since the sheaves are torsion, hence on the E^n terms, shows that the left map induced by ϕ is a quasi-isomorphism, hence an isomorphism in the derived category. Thus the proof for $F = j_!(\mathbb{Z}/n\mathbb{Z})$ is done.

Now we have to prove this for $F = j_!\mathbb{Z}$. We have the following map of triangles in $D(\mathbb{Z})$:

$$\begin{array}{ccccccc} M^{\cdot} & \xrightarrow{n} & M^{\cdot} & \to & M^{\cdot}_n & \to & M^{\cdot}[1] \\ \downarrow^{g} & & \downarrow^{g} & & \downarrow^{g_n} & & \downarrow^{g} \\ N^{\cdot} & \xrightarrow{n} & N^{\cdot} & \to & N^{\cdot}_n & \to & N^{\cdot}[1] \end{array}$$

where $M^{\cdot} = \mathrm{RHom}_{\mathcal{W},X}(j_!\mathbb{Z}, \mathbb{G}_m)$, $N^{\cdot} = \mathrm{RHom}_{\mathbb{Z}}(R\Gamma_{\mathcal{W},X} j_!\mathbb{Z}, \mathbb{Z}[-2])$, $M^{\cdot}_n$ and $N^{\cdot}_n$ are the same with $j_!\mathbb{Z}$ replaced by $j_!(\mathbb{Z}/n\mathbb{Z})$, $g = \kappa_{j_!\mathbb{Z}}$, and $g_n = \kappa_{j_!(\mathbb{Z}/n\mathbb{Z})}$ is a quasi-isomorphism. It is an easy fact from derived categories that, in this scenario, if the homology groups of $M^{\cdot}$ and $N^{\cdot}$ are finitely generated then g is a quasi-isomorphism.

And indeed they are, because $\mathrm{RHom}_{\mathcal{W},X}(j_!\mathbb{Z}, (\mathbb{G}_m)_X) \cong \mathrm{RHom}_{\mathcal{W},U}(\mathbb{Z}, (\mathbb{G}_m)_U) \cong R\Gamma_{\mathcal{W},U}(\mathbb{G}_m)$, whose cohomology groups are finitely generated and zero for large i by theorem 1. [Details: j^* has the exact left adjoint $j_!$, so j^* takes injectives to injectives. And j^* is exact, so it carries a resolution of $\mathbb{G}_{m,X}$ to a resolution of $j^*(\mathbb{G}_m)_X = (\mathbb{G}_m)_U$. $\mathrm{Hom}_U(\mathbb{Z}, (\mathbb{G}_m)_U) \cong \Gamma_U(\mathbb{G}_m)$ - same proof as usual in algebraic geometry - implies $\mathrm{RHom}_U(\mathbb{Z}, (\mathbb{G}_m)_U) \cong R\Gamma_U(\mathbb{G}_m)$.]

$\square$

4 Zeta Functions

Let Γ_0 denote the subgroup $\mathbb{Z} \subseteq \hat{\mathbb{Z}}$ generated by the Frobenius automorphism. Then $H^1(k, \mathbb{Z}) = H^1(\Gamma_0, \mathbb{Z}) = \mathrm{Hom}(\Gamma_0, \mathbb{Z})$ [The category of étale sheaves on $\bar{k}$ is Ab; the category of Weil-étale sheaves is $\mathbb{Z}[\mathbb{Z}]$-modules]. Let $\theta \in H^1(k, \mathbb{Z})$ represent the element corresponding to the homomorphism taking Frob to 1. For any scheme X over k, we also denote by $\theta \in H^1(X, \mathbb{Z})$ the pullback of $\theta \in H^1(k, \mathbb{Z})$. For any Weil-étale sheaf F there is a natural pairing $F \otimes \mathbb{Z} \to F$ induced by $x \otimes n \mapsto nx$. This pairing in turn induces a map $\cup\theta : H^i_{\mathcal{W}}(X, F) \to H^{i+1}_{\mathcal{W}}(X, F)$. $\theta \cup \theta = 0$ [since it is induced from the pullback of $\theta \cup \theta \in H^2(k, \mathbb{Z}) = 0$] so this makes the cohomology groups $(H^i_{\mathcal{W}}(X, F))$ into a complex. Let $h^i_{\mathcal{W}}(X, F)$ be the homology groups of this complex.

Conjecture 1. *U = a quasi-projective variety over k, $j : U \to X$ an open dense immersion of U in a projective variety X. If U is smooth or U is a curve then:*

(1) The cohomology groups $H^i_{\mathcal{W}}(X, j_!\mathbb{Z})$ are finitely generated abelian groups which are zero for large i and are independent of j and X.

(2) Let r_i be the rank of $H^i_{\mathcal{W}}(X, j_!\mathbb{Z})$. Then $\sum(-1)^i r_i = 0$.

(3) The order a_U of the zero of $Z(U, t)$ at $t = 1$ is $\sum(-1)^i i r_i$.

(4) The homology groups $h^i_{\mathcal{W}}(X, j_!\mathbb{Z})$ are finite.

(5) $\lim_{t\to 1} Z(U, t)(1 - t)^{-a_U} = \pm\chi(X, j_!\mathbb{Z}) = \prod_i |h^i(X, j_!\mathbb{Z})|^{(-1)^i}$.

where $Z(U, t) = \exp(\sum_{r=1}^{\infty} N_r t^r / r)$.

Theorem 3. *The conjecture is true if U is projective and smooth, or if U is a smooth surface with X smooth, or if U is any curve.*

5 Geisser's Definition

Unfortunately, virtually every part of Professor Lichtenbaum's conjecture that was not proven is false, as shown by Geisser and Weibel. So, following the idea of Voevodsky of defining a Grothendieck topology generated by Nisnevich covers and abstract blowups, Geisser made the following definition.

Definition 4. *The* **étale h-topology***, or* **eh-topology** *on a subcategory of the category of schemes is the Grothendieck topology generated by étale coverings and abstract blowups, meaning if we have a cartesian square*

$$\begin{array}{ccc} Z' = Z \times_X X' & \xrightarrow{i'} & X' \\ \downarrow^{f'} & & \downarrow^{f} \\ Z & \xrightarrow{i} & X \end{array}$$

where f is proper, i is a closed embedding, and f induces an isomorphism $X' \smallsetminus Z' \xrightarrow{\sim} X \smallsetminus Z$, then $\{X' \xrightarrow{f} X, Z \xrightarrow{i} X\}$ is a covering.

Voevodsky's topology has better properties for singular schemes than Nisnevich's topology, and likewise the eh-topology has better properties than the etale topology. Following are some examples of covers in this topology:

(1) A scheme X is covered by its irreducible components.

(2) $X^{\mathrm{red}} \to X$ is a covering.

(3) For a blowup X' of X with center Z, $\{X' \to X, Z \to X\}$ is a covering.

(4) A proper morphism $p : X' \to X$ such that $\forall x \in X\ \exists y \in p^{-1}(x) | k(y) \cong k(x)$ [they have the same residue field] is a covering, called a **proper eh-covering**.

Proposition 1. *Every eh-covering of X has a refinement of the form $\{U_i \to X' \to X\}_{i\in I}$ where $\{U_i \to X\}_{i\in I}$ is an étale cover and $X' \to X$ is a proper eh-covering.*

Definition 5. *A* **Weil eh-sheaf** *F on $\mathrm{Sch}/\mathbb{F}_q$ is an eh-sheaf on $\mathrm{Sch}/\bar{\mathbb{F}}_q$ together with a $\mathbb{Z}$-action. Weil-eh cohomology is defined as before by the derived functors of $F \mapsto F(\bar{X})^{\mathbb{Z}}$.*

With this definition, all the corresponding parts of Professor Lichtenbaum's conjecture are true.

6 Details

The following commutative diagram of functors induces the map of spectral sequences:

$$\begin{array}{ccccc} \tilde{X}_{\text{ét}} & \to & \tilde{k}_{\text{ét}} & \to & \hat{\mathbb{Z}}\text{-mod} \\ \downarrow^{\phi} & & & & \downarrow \\ \mathrm{Sh}_{\mathcal{W},X} & \to & \mathbb{Z}\text{-mod} & \to & Ab \end{array}$$

Theorem 4. *If X is connected, then*

$$H^q_{\mathcal{W}}(X,\mathbb{Z}) = \begin{cases} \mathbb{Z}, & q = 0,1 \\ H^2_{\text{ét}}(X,\mathbb{Z})/(\mathbb{Q}/\mathbb{Z}), & q = 2 \\ H^q_{\text{ét}}(X,\mathbb{Z}), & q \geq 3 \end{cases}$$

Otherwise, X is a finite disjoint union of schemes. In either case, all these groups are finite.

The étale cohomology is finite, 0 for $q = 1$ and $q >> 0$, $\mathbb{Q}/\mathbb{Z}\oplus$ (finite group) for $q = 2$, and $\mathbb{Z}$ for $q = 0$. [Milne's paper, *Values of Zeta Functions of Varieties over Finite Fields*]

Theorem 5. *$U =$ smooth d-dimensional quasi-projective variety over $k, d \leq 2$. Let $j : U \to X$ be the open immersion into a resolution of singularities X for U (i.e. X is smooth projective). Then $H^q_{\mathcal{W}}(X, j_!\mathbb{Z})$ is finitely generated for all q, zero for q large, and independent of the choices of j and X.*

Key idea used twice in the proof: suppose $j : U \to X$ and $j' : U \to X'$ are two such immersions. Replace X' by the closure of the image of U in $X \times X'$ induced by j and j' (or by the smooth projectivization of the latter). Then there is a map $\pi : X' \to X$ such that $\pi \circ j' = j$, and it turns out $\pi_* j'_!\mathbb{Z} = j_!\mathbb{Z}$ and you go from there.