Introduction to the Cohomology of Topological Groups

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Abstract

For an abstract group G, there is only one "canonical" theory $H^n(G, A)$ of group cohomology for a G-module A. If G is a topological group, however, there are many cohomology theories $H^n(G, A)$ for a topological G-module A. We will discuss some of these and talk about when they give the same results. Some of the topics discussed will be Yoneda Ext's, Grothendieck topologies, cohomology of profinite groups, and applications to number theory. No prerequisites are necessary except basic category theory.

1 Introduction

For an abstract group G, there is one "canonical" cohomology theory $(H^n(G, A))_{n=0}^{\infty}$ for G-modules A, defined by $H^n(G, A) = \mathrm{Ext}^n(\mathbb{Z}, A)$, where the Ext's are taken in the category G-mod of G-modules. Equivalently, $H^n(G, A)$ is the n-th right-derived functor of the functor from G-mod to Ab, the category of abelian groups, taking A to A^G, the group of points of A that are fixed by G. Note that $\mathrm{Hom}_{G\text{-mod}}(\mathbb{Z}, A) = A^G$ and $\mathrm{Ext}^n(\mathbb{Z}, -)$ are the derived functors of $\mathrm{Hom}(\mathbb{Z}, -)$; this is why the two are equivalent definitions. There are at least two other equivalent definitions. One is via the cohomology of the complex $(C_h^n(G, A), \delta_n^h : C_h^n(G, A) \to C_h^{n+1}(G, A))_{n=0}^{\infty}$ of

homogeneous cochains, where $C_h^n(G,A) = \mathrm{Hom}_G(G^{n+1}, A)$ and

$$\delta_n^h(f)(x_0,\ldots,x_{n+1}) = \sum_{k=0}^{n+1}(-1)^k f(x_0,\ldots,\hat{x}_k,\ldots,x_n)$$

The elements of $C_h^n(G,A)$ are called homogeneous cochains. The other equivalent definition is via the cohomology of the complex $(C^n(G,A), \delta_n : C^n(G,A) \to C^{n+1}(G,A))_{n=0}^{\infty}$ of **inhomogeneous cochains**, where $C^n(G,A)$ is the set of all maps $G^n \to A$, $G^0 = \{*\}$ is a point, and

$$\begin{aligned}(\delta_n(f))(x_0,\ldots,x_n) &= x_0 f(x_1,\ldots,x_n) \\ &+ \sum_{k=1}^{n}(-1)^k f(x_0,\ldots,x_{k-2},x_{k-1}x_k,x_{k+1},\ldots,x_{n+1}) + f(x_0,\ldots,x_{n-1})\end{aligned} \tag{1.1}$$

From now on, let G be a topological group. We want a cohomology theory on the category $\mathcal{M}_G$ of topological G-modules A[1] that utilizes the topologies of both G and A. Since $\mathcal{M}_G$ is not abelian, we cannot define derived functors of the functor $A \mapsto A^G$ from $\mathcal{M}_G$ to Ab. But we can almost do so by using Yoneda's Ext's. This is only way of generalizing the usual cohomology, and there are many others, using for example cochain definitions, Grothendieck topologies, and other tools. For other theories, see [Wig73], [HM62], [FW12], [Seg70], and [Sta78], among others.

2 Yoneda Ext Definition

In this section we first define quasi-abelian S-categories, as Yoneda did, and then use them to define cohomology theories for topological groups.

A morphism $f : A \to B$ in an additive category is **proper**, or **strict**, if the natural map $\mathrm{coker}(\ker f) \to \ker(\mathrm{coker}\, f)$ is an isomorphism and, in particular, these kernels and cokernels exist. In this case, the object associated to $\mathrm{coker}(\ker f)$ is called the **image** $\mathrm{Im}(f)$ of f[2]. A sequence $A \xrightarrow{f} B \xrightarrow{g} C$ is **exact** if f and g are proper, $g \circ f = 0$, and the natural map $\mathrm{Im}(f) \to \ker(g)$

[1] A topological G-module A is both a G-module and a topological abelian group such that the action $G \times A \to A$ is continuous.

[2] Technically, this object is not unique, but only unique up to isomorphism.

is an isomorphism.

An S**-category** $(\mathcal{C}, S)$ is an additive category $\mathcal{C}$ together with a class S of morphisms of $\mathcal{C}$ such that:

(S1) All isomorphisms are in S, and all maps in S are proper.

(S2) For any two morphisms $f : A \to B, g : C \to D$ in S, the morphism $f \oplus g : A \oplus C \to B \oplus D$ is in S.

(S3) If $\phi \in S$ then $\ker \phi \in S$ and $\operatorname{coker} \phi \in S$.

(S4) Any $f \in S$ can be written $f = me$, where $m, e \in S$, $\ker m = 0$, $\operatorname{coker} e = 0$, and any such composition me is in S.

The class $P(\mathcal{C})$ of all proper morphisms satisfies all these properties and so is the largest possible class S for $\mathcal{C}$. An S-category is **quasi-abelian** if it satisfies the following four conditions.

(Q0) A composition of epimorphisms in S is an epimorphism in S.

(Q0*) A composition of monomorphisms in S is a monomorphism in S.

(Q2) Every pullback of an epimorphism in S exists and is (an epimorphism) in S.

(Q2*) Every pushout of a monomorphism in S exists and is (a monomorphism) in S.

A category $\mathcal{C}$ is **quasi-abelian** if $(\mathcal{C}, P(\mathcal{C}))$ is quasi-abelian.

It turns out that the category $\mathcal{M}_G$ of all topological G-modules and continuous G-equivariant maps is quasi-abelian. In fact, every morphism $f : A \to B$ in $\mathcal{M}_G$ has a kernel and a cokernel, but f is proper if and only if the induced map $A \to \operatorname{Im}(A)$ is an open map, when $\operatorname{Im}(A)$ is considered as a subspace of B.

For any quasi-abelian S-category $(\mathcal{C}, S)$, we can define $\operatorname{Ext}^n_{\mathcal{C},S}(A, B)$ to be the class of extensions (long exact sequences)

$$X : 0 \to B \to E_n \to \cdots \to E_1 \to A \to 0$$

modulo the equivalence relation generated by commutative diagrams

$$\begin{array}{ccccccccccccccc} X: & 0 & \to & B & \to & E_n & \to & \cdots & \to & E_1 & \to & A & \to & 0 \\ & & & \| & & \downarrow & & & & \downarrow & & \| & & \\ X': & 0 & \to & B & \to & E'_n & \to & \cdots & \to & E'_1 & \to & A & \to & 0 \end{array}$$

It turns out that

1. $\mathrm{Ext}^n_{C,S}(A, B)$ is an abelian group (see [Yon60, p. 537] for more details.);
2. for any $Z \in C$, any short exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ in S (i.e. with $f, g \in S$) gives a long exact sequence
$$0 \to \mathrm{Hom}_C(Z, A) \to \mathrm{Hom}_C(Z, B) \to \mathrm{Hom}_C(Z, C) \to \mathrm{Ext}^1_{C,S}(Z, A) \to \cdots$$
3. and there is a universality property: for any collection of functors $(h^n)_{n=0}^{\infty}$ from C to Ab and any natural transformation $\eta_0 : h^0 \to \mathrm{Hom}_C(Z, -)$, if any short exact sequence as above in S gives a long exact sequence
$$h^0(A) \to h^0(B) \to h^0(C) \to h^1(A) \to \cdots$$
then there are unique natural transformations $\eta_n : \mathrm{Ext}^n_{C,S}(Z, -) \to h^n$ that extend η_0.[3]

For any class S of morphisms of $\mathcal{M}_G$ such that $(\mathcal{M}_G, S)$ is quasi-abelian, we can then define a cohomology theory by $H^n(G, A) = \mathrm{Ext}^n_{\mathcal{M}_G,S}(\mathbb{Z}, A)$, where $\mathbb{Z}$ has trivial G-action and discrete topology.

3 Cochain Definitions

One way to define cohomology theories $H^n(G, A)$ is via inhomogeneous cochains[4]: we set $C^n(G, A) = \mathrm{Mor}_C(G^n, A)$, $n = 0, 1, \ldots$, where $G^0 = \{*\}$ is a point and C is one of at least three categories:

1. C^G_c, the category of G-spaces and continuous maps;

[3] This means for any short exact sequence in C, the induced diagram of long exact sequences is commuative.

[4] We could also define each cochain theory by using homogeneous cochains instead.

2. $\mathcal{C}_m^G$, the category of G-spaces and measurable[5] maps;

3. $\mathcal{C}_{lcm}^G$, the category of pointed G-spaces (X, x) measurable maps $f : (X, x) \to (Y, y)$ which are **locally continuous**, i.e. there is an open set U of x such that $f|U$ is continuous (with $f(x) = y$).[6]

Then we define the inhomogeneous coboundary operator δ_n just as before by equation (1.1). These categories give cohomology theories

1. $H_c^n(G, A)$, called the **continuous cochain theory**;

2. $H_m^n(G, A)$, called the **measurable cochain theory**;

3. $H_{lcm}^n(G, A)$, called the **locally continuous measurable cochain theory**

respectively, and these are in general not the same, but[7]

$$H_c^n(G, A) \subseteq H_{lcm}^n(G, A) \subseteq H_m^n(G, A)$$

There are classes S of morphisms in $\mathcal{M}_G$ corresponding to the various cohomology theories. These classes S are defined by insisting that a short exact sequence

$$0 \to A \to B \xrightarrow{\phi} C \to 0 \tag{3.1}$$

is in S if and only if ϕ has a section in the corresponding category $\mathcal{C}$. These short exact sequences yield long exact sequences

$$\cdots \to H^n(G, A) \to H^n(G, B) \to H^n(G, C) \to H^{n+1}(G, A) \to \cdots$$

for the corresponding cohomology theories. For example, if ϕ has a continuous section, there is a long exact sequence on cohomology (and not otherwise, in general, as the following example shows).

[5]A map $f : X \to Y$ of topological spaces is (Borel-)measurable if the preimage $f^{-1}(U)$ of every open set U in Y is in the Borel σ-algebra, the σ-algebra generated by the open sets in X.

[6]Technically, in this case we should say $C^n(G, A) = \mathrm{Mor}_{\mathcal{C}_{lcm}^G}((G^n, (1, \ldots, 1)), (A, 0))$.

[7]It is not immediately obvious why $H_c^n(G, A) \subseteq H_{lcm}^n(G, A)$, but with a simple argument using a short exact sequence of complexes, one can show that $H_c^n(G, A)$ would be the same if one replaced $\mathcal{C}_G$ by the category of pointed G-spaces and continuous G-equivariant maps.

Example computation. If G is connected and A is discrete, then $C_c^n(G, A)$ consists of constant maps, so $H_c^n(G, A) = 0$ for $n > 0$. We leave it as an easy exercise to the reader to show that the action of G on A must be trivial, using the continuity of $G \times A \to A$. Now consider the exact sequence of trivial $\mathcal{S}^1$-modules

$$0 \to \mathbb{Z} \to \mathbb{R} \xrightarrow{\pi} \mathcal{S}^1 \to 0 \tag{3.2}$$

Note that π has no continuous section, so theorem 3.2 does not imply there is a long exact sequence on cohomology corresponding to (3.2). Indeed, the sequence

$$H^1(\mathcal{S}^1, \mathbb{Z}) \to H^1(\mathcal{S}^1, \mathbb{R}) \to H^1(\mathcal{S}^1, \mathcal{S}^1) \to H^2(\mathcal{S}^1, \mathbb{Z})$$

is not exact. By the above remark, $H^1(\mathcal{S}^1, \mathbb{Z}) = H^2(\mathcal{S}^1, \mathbb{Z}) = 0$, and $H^1(\mathcal{S}^1, \mathbb{R}) = \mathrm{Hom}_{\mathrm{cont}}(\mathcal{S}^1, \mathbb{R}) = 0$ because there are no nontrivial continuous homomorphisms from $\mathcal{S}^1$ to $\mathbb{R}$. On the other hand, $H^1(\mathcal{S}^1, \mathcal{S}^1) = \mathrm{Hom}_{\mathrm{cont}}(\mathcal{S}^1, \mathcal{S}^1) \neq 0$.

It should be noted that if we work in the category $\mathcal{M}_G^p$ of complete metric second-countable Hausdorff[8] G-modules, then any epimorphism $B \xrightarrow{\phi} C$ has a measurable section. It is easy to see that $\mathcal{M}_G^p$ is also quasi-abelian. So all exact sequences in $\mathcal{M}_G^p$ yield long exact sequence on cohomology for the measurable cochain theory.

Theorem 3.1. *If G is locally compact, second countable, and Hausdorff, then* $\mathrm{Ext}^n_{\mathcal{M}_G^p, P(\mathcal{M}_G^p)}(\mathbb{Z}, A) = H_m^n(G, A)$.

Theorem 3.2. *If G is locally compact, then* $\mathrm{Ext}^n_{\mathcal{M}_G, S}(\mathbb{Z}, A) = H_c^n(G, A)$, *where S is the class of morphisms corresponding to the class C of short exact sequences (3.1) where ϕ has a continuous section.*

At this point it is not known whether or not such a statement for $H_{lcm}^n(G, A)$.

Application. In class field theory, one computes the cohomology of profinite groups G, with the profinite topology, by using $H_c^n(G, A)$ for discrete G-modules A. Note that a topological group is profinite if and only if it is compact and totally disconnected.

[8]Such G-modules are called **Polish**.

Theorem 3.3 ([Wig73]). *If G is locally compact, σ-compact*[9]*, and zero-dimensional*[10]*, and A is Polish, then $H^n_c(G, A) = H^n_m(G, A)$ for all n.*

4 Grothendieck Topologies

Definition 4.1. *A* **Grothendieck topology**[11] *$T = (C, \mathrm{Cov}(C))$ is a category C that has pullbacks together with a collection of coverings $\mathrm{Cov}(C)$ that consists of collections $\{A_i \xrightarrow{f_i} A\}$ for each object A in C such that:*

1. *Isomorphisms are coverings, i.e. $\{f : A \to B\}$ is a covering for any isomorphism f.*
2. *A pullback of a covering is a covering: if $\{A_i \to A\}$ is a covering and $f : B \to A$ is any map in C, then $\{A_i \times_A B \to B\}$ is a covering.*
3. *A covering of a covering is a covering: if $\{A_i \to A\}$ is a covering and for each i we have a covering $\{A_{i,j} \to A_i\}$ then $\{A_{i,j} \to A\}$ is a covering.*

Definition 4.2. *A* **sheaf** *on a Grothendieck topology $T = (C, \mathrm{Cov}(C))$ is a contravariant functor from C to* Ab *such that for every covering $\{A_i \to A\}$ in $\mathrm{Cov}(C)$ the induced diagram*

$$F(A) \to \prod F(A_i) \rightrightarrows \prod F(A_i \times_A A_j)$$

is exact.

A theorem of Grothendieck says that the category of sheaves on any Grothendieck topology has enough injectives, so one can define cohomology $H^n(T, X, F)$ for any object X in C and any sheaf F on T as the n-th right-derived functor of $F \mapsto F(X)$.

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[9] A space is σ-compact if it is the union of countably many compact subsets.

[10] The concept of dimension here is that of Lebesgue dimension: we say that for a space X, $\dim X \leq n$ if any covering $\{U_i\}$ of X has a refinement $\{V_i\}$ such that for all $x \in X$, $x \in V_i$ for at most $n + 1$ indices i. The dimension m of X is the smallest m such that $\dim X \leq m$ but not $\dim X \leq m - 1$ (if no such m exist, then $\dim X = \infty$).

[11] Actually, this is what Grotendieck termed a "pretopology"; a pretopology generates a topology, a notion which is defined using sieves. The category of sheaves for a pretopology is the same as that for the topology generated by it, and that is all we really care about.

Another equivalent way of computing the standard group cohomology $H^n(G, A)$ is to let G-set be the category of G-sets and G-equivariant maps, and define a topology T on G-set by letting $\{X_i \xrightarrow{f_i} X\}$ be a covering if $X = \bigcup f_i(X_i)$. For any G-module A we have the sheaf $\tilde{A} = \mathrm{Hom}_{G\text{-set}}(-, A)$, and it turns out $H^n(G, A) = H^n(T, \{*\}, \tilde{A})$.

We can generalize this construction in many ways to get cohomology theories of topological groups. For example, we can let $C^G_{c,eq}$ be the category of G-spaces and continuous G-equivariant maps. We can define a system of coverings on $C^G_{c,eq}$ in a number of different ways to get topologies T on $C^G_{c,eq}$ and then define a cohomology theory by $H^n(G, A) = H^n(T, \{*\}, \tilde{A})$, where $\tilde{A} = \mathrm{Hom}_{C^G_{c,eq}}(-, A)$, provided $\tilde{A}$ is a sheaf. If $\tilde{A}$ is a sheaf for all G-modules A, the topology T is called **subcanonical**.

Some such ways of defining T may be more useful than others. Two important cases are:

1. Let $\{X_i \xrightarrow{f_i} X\}_{i \in I}$ be a covering if for all $x \in X$ there is $i \in I$, a neighborhood U of x in X, and a continuous map $s : U \to X_i$ such that $f_i \circ s = \mathrm{id}_U$. The induced topology is subcanonical, and the cohomology $H^n_W(G, A)$ is the same as the one Wigner defined in [Wig73]; this was proven by Lichtenbaum in [Lic09] and then used to construct his Weil-étale topology for number fields, which led to conjectures about special values of the zeta-function. For any short exact sequence (3.1) such that $\{B \xrightarrow{\phi} C\}$ is a covering in T, we get a long exact sequence on cohomology.

2. Let the only coverings for $T = T_c$ be $\{X \xrightarrow{f} Y\}$ such that there is a continuous (not necessarily G-equivariant) section s of f. I proved $H^n(T_c, \{*\}, \tilde{A}) = H^n_c(G, A)$ for all n.

3. Let T_m be the topology on the category $C^G_{m,eq}$ of G-spaces and measurable G-equivariant maps where the coverings are $\{X \xrightarrow{f} Y\}$ such that there is a measurable section s of f. I proved $H^n(T_m, \{*\}, \tilde{A}) = H^n_m(G, A)$.

4. Let T_{lcm} be the topology on the category $C^G_{lcm,eq}$ of pointed G-spaces and measurable locally continuous G-equivariant maps where the

coverings are $\{(X,x) \xrightarrow{f} (Y,y)\}$ such that f has a locally continuous measurable sections s. I proved $H^n(T_{lcm}, \{*\}, \tilde{A}) = H^n_{lcm}(G,A)$.

Theorem 4.3. *[Wig73] If G is locally compact, σ-compact, finite-dimensional, and A is Polish and has Wigner's "property F", then $H^n_W(G,A) = H^n_m(G,A)$.*

5 $H^2(G,A)$

It is well-known that, for the standard theory of group cohomology, given a G-module A, $H^2(G,A)$ classifies the extensions

$$1 \to A \to E \xrightarrow{\psi} G \to 1 \tag{5.1}$$

of abstract groups with the fixed action of G (i.e. such that the action of G on A by conjugation is the same as the original action of G on A given in the definition of A as a G-module). Note that this does not give an action of G on E, so this is not an extension of G-sets.

Theorem 5.1. *[Hu52] Given a topological G-module A, $H^2_c(G,A)$ classifies the extensions (5.1) of topological groups with the fixed action of G on A such that the map ψ has a continuous section.*

Theorem 5.2. *[Moo76] If G is locally compact, Hausdorff, and second countable, and A is a second countable topological G-module whose topology is given by a complete metric, then $H^2_m(G,A)$ classifies all extensions (5.1) of topological groups with the fixed action of G on A.*

Moore used this theorem as justification that his cohomology $H^n_m(G,A)$ may be the right generalization of the usual group cohomology theory to topological groups and G-modules.

Theorem 5.3. *With the assumptions in theorem 5.2, $H^2_{lcm}(G,A)$ classifies all extensions (5.1) of topological groups with the fixed action of G on A such that ψ has a local section.*

It is not known whether or not $H^2(G,A)$ for Lichtenbaum's cohomology classifies the extensions (5.1) such that ψ has local sections. If this is true, then it would seem that there should be an equivalent way of defining the cohomology via cochains, since $H^2(G,A)$ is so closely related to the cochain definition.

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