Projective Geometry in a Plane

Fundamental Concepts

Undefined Concepts: Point, line, and incidence

Axiom 1. Any two points P, Q lie on exactly one line, denoted PQ.
Axiom 2. Any two lines l, m intersect in at least one point, denoted $l \bullet m$.

Definition. A **quadrangle** is a set of four points, no three of which are collinear.

Axiom 3. A quadrangle exists.
Axiom 4. If $PQRS$ is a quadrangle, then its **diagonal points** $PQ \bullet RS$, $PR \bullet QS$, and $PS \bullet QR$ are not collinear.

Definition. A **range** on l is a set of points lying on the line l.
Definition. A **pencil** at P is a set of lines passing through the point P.
Definition. An **elementary projectivity** is the correspondence between lines in a pencil and points in a range given by incidence:

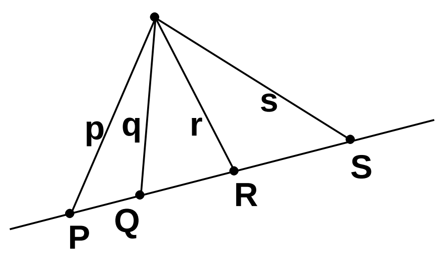

Figure 1. An elementary projectivity.

Definition. A **projectivity** is a finite composition of elementary projectivities, written as $PQRS \;\overline{\wedge}\; P'Q'R'S'$, $PQRS \;\overline{\wedge}\; pqrs$, $pqrs \;\overline{\wedge}\; p'q'r's'$, or $pqrs \;\overline{\wedge}\; PQRS$.

Axiom 5. If a projectivity fixes three points on a line, it fixes every point on the line.

Definition. A **perspectivity,** denoted $\overline{\overline{\wedge}}$, is a composition of two projectivities. If lines in a pencil at P relate the two ranges in a perspectivity (see Figure 2), the ranges are said to be **perspective from center *P***. If the ranges lie on lines l and m, this may also be called a projection of points on l to points on m from P (or vice-versa)

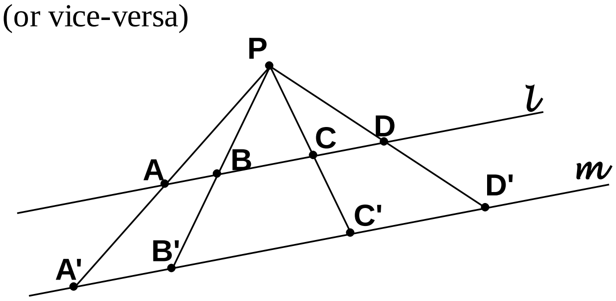

Figure 2. A perspectivity of ranges.

Theorem 1 (Fundamental Theorem of Projective Geometry). Given three collinear points A, B, C (or concurrent lines a, b, c) and the corresponding three collinear points A', B', C' (or concurrent lines a', b', c'), there is a unique projectivity relating ABC (or abc) to $A'B'C'$ (or $a'b'c'$).

Exercise. Prove that the axioms are dual in the concepts of a point and a line, i.e. if we interchange the words "point" and "line" and their respective synonyms for incidence, etc., the new statement is also true. This is the notion of duality in projective geometry.

Two Basic Theorems

Lemma. The projectivity $ABC \barwedge A'B'C'$, with A, B on line l and A', B', C' on line m, and $C = l \bullet m$, is a perspectivity if and only if $C = C'$.

Proof. If $C = C'$ then the perspectivity from $AB \bullet A'B'$ taking ABC to $A'B'C'$ is the unique projectivity relating these points. If the projectivity is a perspectivity, then it must fix the intersection of the two lines.

Theorem 2 (Desargues's Theorem). Given two triangles PQR and $P'Q'R'$, the joins PP', QQ', RR' of the three pairs of vertices are concurrent if and only if the intersections $QR \bullet Q'R'$, $RP \bullet R'P'$, and $PQ \bullet P'Q'$ of the corresponding sides are collinear.

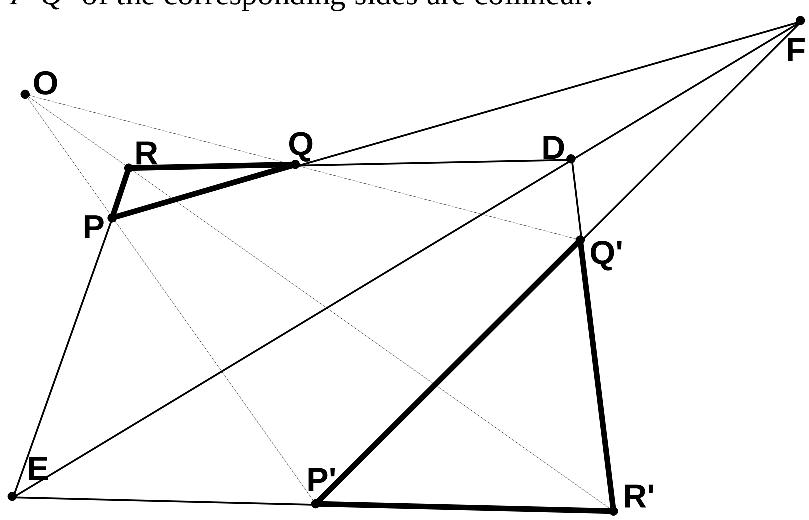

Figure 3. The Desargues Configuration.

Proof. Suppose PP', QQ', and RR' all go through a point O. We must show that the points $D = QR \bullet Q'R'$, $E = RP \bullet R'P'$, and $F = PQ \bullet P'Q'$ are collinear. Let $A = PP' \bullet DE$, $B = QQ' \bullet DE$, and $C = RR' \bullet DE$. Then $PAP'O \doublebarwedge RCR'O$ is a perspectivity from E, and $RCR'O \doublebarwedge QBQ'O$ is a perspectivity from D, so there is a projectivity $PAP'O \barwedge QBQ'O$, which fixes the intersection of the two lines. By the lemma, this is a perspectivity from $F = PQ \bullet P'Q'$, so D, E, and F are collinear. For the converse, apply what we just proved to the triangles $PP'E$ and $QQ'D$.

Note the symmetries in Figure 3: there are 10 lines each passing through 3 points, and 10 points each lying on 3 lines (this is called a 10_3 configuration); the pentagons *EPQQ'R'* and *RDFP'O* are mutually inscribed; and the quadrilateral *(QR)(RP)(PQ)(DE)* and quadrangle *OP'Q'R'* are mutually inscribed.

Theorem 3 (Pappus's Theorem). If the points *A*, *B*, *C* are collinear and *A'*, *B'*, *C'* are collinear, then the "cross-joins" $L = BC'\bullet B'C$, $M = CA'\bullet C'A$, and $N = AB'\bullet A'B$ are collinear.

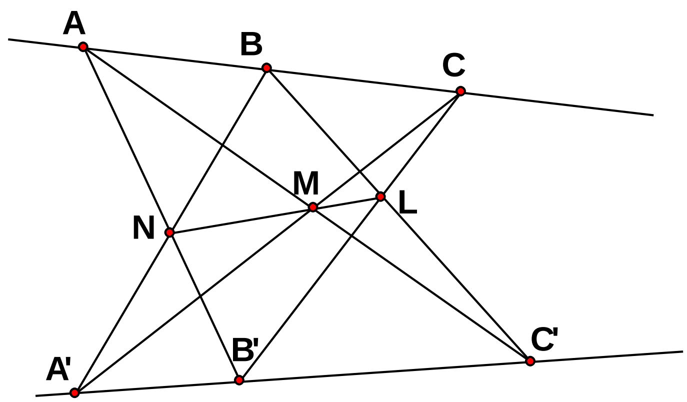

Figure 4. Pappus's theorem.

Proof. Let $I = AB\bullet A'B'$, $J = AB'\bullet CA'$, and $K = AC'\bullet CB'$. Then $ANJB' \overline{\overline{\wedge}} ABCE \overline{\overline{\wedge}} KLCB'$ (first from *A'*, then from *C'*). Since the intersection of the ranges is fixed, this is a projectivity with center $M = AK\bullet JC$, hence *N*, *M*, and *L* are collinear.

Exercises.

1. State (and draw) the dual of Desargues's and Pappus's theorems.
2. What kind of configuration (e.g. 10_3 above) is Figure 4?

Conics

Definition. A **polarity** $\boldsymbol{\pi}$ is a transformation of the plane that takes points to lines and lines to points (so the image of a pencil must be a range, and the image of a range must be a pencil), preserves incidence, has degree 2 (the composition with itself is the identity), and such that all restrictions of π to pencils and ranges are projectivities.

Notation. If the polarity is understood, the image of a point *X* is the line denoted *x*, and vice versa. In this case, *X* is the **pole** of *x*, and *x* is the **polar** of *X*.

Definition. Two points *X*, *Y* (resp. lines *x*, *y*) are **conjugate** if they are incident with each other's polars (resp. poles), i.e. if *X* lies on *y* (and thus also *Y* lies on *x* since incidence is preserved).
Definition. A **self-conjugate** point (or line) is incident with its image under a polarity.

- A line joining two self-conjugate points cannot be self-conjugate.

- A line cannot contain more than two self-conjugate points.
- If X is a self-conjugate point, every line except x contains another self-conjugate point.

Definition. A polarity is called **hyperbolic** if a self-conjugate point exists and **elliptic** otherwise.
Definition. A **conic** is the set of self-conjugate points and lines (the **tangents**) of a polarity.
Definition. A point not on a conic is called **exterior** if it lies on two self-conjugate lines, and **interior** if it lies on none.

Theorem 4 (Steiner's Definition). A **conic** is the set of intersections of two pencils of lines that are projectively, but not perspectively, related.

Note. At this point, the betweenness axioms (the same ones apply for Euclidean geometry and hyperbolic geometry) could be introduced, as well as their consequences in regard to conics, and the continuity axiom.

Affine geometry: To get affine geometry from projective geometry, select a line l_∞ and call it the **line at infinity**. Then call two lines **parallel** if they intersect at a point on l_∞. A **parallelogram** consists of four lines, two pairs of which are parallel. Two segments (which can be defined by using the betweenness axioms) have the same **length** if they are the opposite sides of a parallelogram. So it is possible to define distance along parallel lines only, and also area, in affine geometry.

Definition. An **elliptic involution** is a projectivity on a line that has period 2 and leaves no points fixed.

Euclidean geometry: To get Euclidean geometry from affine geometry, pick an elliptic involution on l_∞ and call it the **absolute involution**. Two lines are then said to be **perpendicular** if their intersections with l_∞ are images of each other under the absolute involution.

Hyperbolic geometry: To get hyperbolic geometry from projective geometry with betweenness axioms, pick a conic corresponding to a hyperbolic polarity (e.g. in real projective geometry, this may correspond to the conic $x^2+y^2=1$). The points in the hyperbolic plane are the interior points of the conic. Two lines are **parallel** if they do not meet at an interior point of the conic. They are **ultraparallel** if they intersect at an exterior point of the conic and **asymptotic parallels** if they meet at a point on the conic (hence each line has two asymptotic parallels through any point not on it). Two lines are perpendicular if they are conjugate under the polarity.

Elliptic geometry: To get elliptic geometry from projective geometry, pick an elliptic polarity. All lines intersect in elliptic geometry, and two lines are **perpendicular** if they are, again, conjugate under the polarity.

Reference

[pg] Coxeter, Harold Scott Macdonald. *Projective Geometry, 2nd edition.* Toronto: University of Toronto Press, 1974.