Teaching with a Smile

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Outline

Yin-Yang of Mathematics

Yang	Yin
Logic	Creativity
Working hard to solve the problem	Letting the solution come to you
Right or wrong	Multiple approaches
Hard and dry	Fun and beautiful

Attitude

Easy Problems

Easy problems are there for students to practice and feel good about their skill set

Hard Problems

Hard problems are the ones they don't know how to solve immediately. These are the fun ones where they get to be creative and discover something new for themselves.

Puzzles

Examples

Sudoku, Calcudoku (Kenken), Kakuro, Hidato, Nurikabe, Skyscrapers, Tents, etc.

Benefits

- Show students mathematics is much broader and can be much more fun than they thought
- Engage students to pay attention better throughout the whole class
- Allow students to participate in class more comfortably
- Introduce students to NP-complete problems

Other Fun Stuff

Puzzles are enough to engage students and open their minds about math a little, but we can do more by introducing other topics:

Topics

- Induction
- Combinatorics
- Rational / Irrational Numbers
- Magic squares
- Golden ratio
- Modular Arithmetic
- MANY more

Induction

- Prove the formulas used for Riemann sums:

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}, \ \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}, \ \sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$

- Fibonacci numbers
- Find the n-th derivative of a function, plug in c to get Taylor series centered at c

Combinatorics Part 1: Jumbles

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Combinatorics Part 1: Jumbles

NOODLE

Combinatorics Part 2: Formula

Number of permutations of a word =

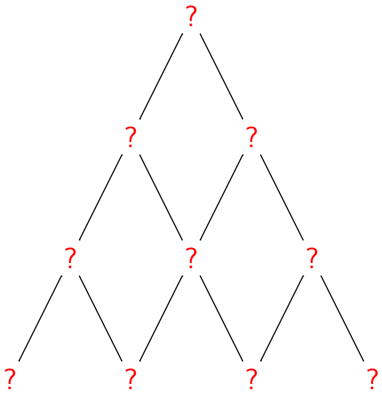
$$\frac{\text{(length of word)!}}{\prod_{\text{letter}=A}^{Z}\text{(number of times letter appears in word)!}}$$

Combinatorics Part 3: Permutations

$$\binom{n}{r} = \left(\text{number of permutations of } \underbrace{\mathrm{AA}\ldots\mathrm{A}}_{r}\,\underbrace{\mathrm{BB}\ldots\mathrm{B}}_{n-r}\right) = \frac{n!}{r!(n-r)!}$$

Combinatorics Part 4: Pascal's Triangle

How many ways to get from the top ? to the bottom ones without going up?



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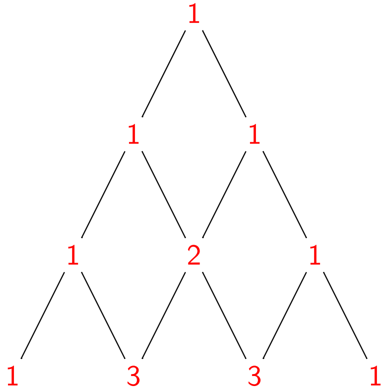

Combinatorics Part 5: Gathering What We've Learned

The number of ways to get to the r-th spot in the n-th row (n and r start at 0) is the number of permutations of

$$\underbrace{\mathrm{RR}\ldots\mathrm{R}}_{r}\underbrace{\mathrm{LL}\ldots\mathrm{L}}_{n-r}$$

which is

$$\frac{n!}{r!(n-r)!} = \binom{n}{r}$$

Combinatorics Part 5: Gathering What We've Learned

We've proven combinatorially that

$$\binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1}$$

Exercise. Prove it algebraically!

Combinatorics Part 6: Binomial Coefficients

$$(x + y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r}$$

because the number of $x^r y^{n-r}$ terms in the expansion is the number of permutations of

$$\underbrace{xx\cdots x}_{r}\underbrace{yy\cdots y}_{n-r}$$

Combinatorics Part 7: Application to Calculus

Let u and v be functions. The n-th derivative of uv is:

$$(uv)^{(n)} = \sum_{r=0}^{n} \binom{n}{r} u^{(r)} y^{(n-r)}$$

$$(xe^x)^{(n)} = \binom{n}{0} x^{(0)} (e^x)^{(n)} + \binom{n}{1} x^{(1)} (e^x)^{(n-1)} = xe^x + ne^x$$

Later find the Taylor series for xe^x centered at 2.

Rational / Irrational Numbers

Question

Is an irrational power of an irrational number always irrational?

Hint: $(\sqrt{2})^2 = 2$.

Rational / Irrational Numbers

Question

Is an irrational power of an irrational number always irrational?

No: $(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = 2$.

Caution: Don't Get Overzealous!

Don't feed your students a whole chicken with Thanksgiving dinner! I.e. don't do too many puzzles!

Projects

Definition

A 5-7 page paper (double-spaced) on a topic the student chooses, which uses calculus in some way

Grading

- Relevance - 25%
- Mathematics - 50%
- Effort - 20%
- Enthusiasm - 5%
- Typed in LaTeX - 5% Extra Credit
- Presentation - 10% Extra Credit

Benefits

- Teaches students what mathematics is all about
- Exposes students to a new idea in depth
- Allows students to explore mathematics in their own way, as opposed to being told how to
- Begins to teach students to think like a mathematician

One Student's Comment

"The final project was so so great! I did my project on the concept and idea of infinity and I honestly had never considered math in a way outside of problems, equations or graphs. However, the project provided a chance for me to do research and delve further into an interesting and different perspective on math than I had ever really considered. I think that doing a final project in a calculus II class allowed everyone in the class to look into topics of interest that may not otherwise be addressed except for in much higher level theory or conceptual math classes. The final project was a very very cool component of the class. "

Examples of Projects Done

- e is irrational and even transcendental
- History of π
- Bezier curves
- Calculus and economics
- The calculus of car crash testing
- Prime number theorem and the Riemann zeta function: the basics

One Project

Illegal Summations and Mathematically Valid Explanations

$$\sum_{n=1}^{\infty}(-1)^{n-1} = 1 - 1 + 1 - 1 + \cdots = \frac{1}{2}$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}n = 1 - 2 + 3 - 4 + \cdots = \frac{1}{4}$$

$$\sum_{n=1}^{\infty} n = 1 + 2 + 3 + 4 + \cdots = \frac{-1}{12}$$

Illegal Summation

For example,

$$S = \sum_{n=1}^{\infty} (-1)^{n-1} = 1 - 1 + 1 - 1 + \cdots = \frac{1}{2} :$$

$$\begin{array}{rl} S + S = & 1 - 1 + 1 - 1 + 1 - \cdots \\ + & \quad\;\; 1 - 1 + 1 - 1 + \cdots \\ \hline = & 1 \end{array}$$

$$2S = 1 \Rightarrow S = \frac{1}{2}$$

Cesàro Summation

Definition

The Cesàro summation of $\{a_n\}_{n=1}^{\infty}$ *is*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} S_k,$$

the limit of the averages of partial sums $S_k = \sum_{i=1}^{k} a_i$.

- The Cesàro summation of a convergent series is the usual sum of the series.
- The Cesàro summation of $\{(-1)^{n-1}\}_{n=1}^{\infty}$ is $\dfrac{1}{2}$ because half of the partial sums are 1 and half are 0.

Other Explanations

- The Euler summation of $\{(-1)^{n-1}n\}_{n=1}^{\infty}$ is $\dfrac{1}{4}$
- The Riemann zeta function at -1 is

$$\zeta(-1) = \sum_{n=1}^{\infty} n = \frac{-1}{12}$$

Other Projects to Consider

- Explain how to solve one type of differential equation
- DeMoivre's formula and Chebyshev polynomials
- O-, o- notation and algorithms in computer science
- Write your own program to compute π
- Application of calculus (or math) to another science or area of life
- Hamilton / Euler circuits and the Königsberg bridge problem
- Infinite sets and cardinalities

Thank you so much
for your attention!