Cohomology of Topological Groups And Grothendieck Topologies

by

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Vitæ

Igor Minevich was born in Dnepropetrovsk, Ukraine (now Dnipropetrovs'k) on September 18, 1987. He moved to America on August 22, 1997 and resided in Indianapolis, IN until 2009. Igor received a B.S. in mathematics and minors in physics and computer science in 2009 from Indiana University Purdue University Indianapolis (IUPUI). While at IUPUI, he was supported by the Bepko Scholarship, which covered all tuition. Igor started at Brown University in the Fall of 2009. While at Brown, Igor was supported by two GAANN grants and a dissertation fellowship. He married Natasha Arora on May 28, 2012. He was a recipient of the Mathematics Department Outstanding Teaching Award in the Spring of 2014.

Dedicated to my parents and my beloved wife

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Contents

Introduction

The subject of group cohomology is well-known to number theorists, topologists, and others. There are many different equivalent definitions of group cohomology. The question that has been pondered for many decades now (since at least the 1950's) is, what is the right generalization of group cohomology to a theory $(H^n(G, A))_{n=0}^{\infty}$ for a topological group G and a topological G-module A?

Let us look at the basic properties of group cohomology. $H^0(G, A) = A^G$, the group of elements of A fixed by G. For a short exact sequence of G-modules, we obtain a functorial long exact sequence on cohomology. Group cohomology is universal in the sense of these two properties, that is, if we have another sequence of functors (h^i) that give long exact sequences for short exact sequence of G-modules and a natural transformation $H^0 \to h^0$ then there exist unique natural transformations $H^i \to h^i$ for all i which induce maps of long exact sequences for every short exact sequence of G-modules. $H^1(G, A)$ classifies the crossed homomorphisms $f : G \to A$ satisfying $f(gh) = gf(h) + f(g)$, modulo the principal crossed homomorphisms of the form $f(g) = g \cdot a - a$ for some $a \in A$. $H^2(G, A)$ classifies the extensions of groups $1 \to A \to E \to G \to 1$ of G by A such that the action of G on A by conjugation is the same as the original action of G on A. Certainly, we would want a cohomology theory for topological groups to satisfy these properties, but one can hope for more: since there is topological data on G and A, we want to have topological data on the groups $H^n(G, A)$.

The first generalizations came with W. T. van Est's exploration of the "smooth cochain" theory for Lie groups and its relation with the Lie algebra cohomology [**32**] and Sze-tsen Hu's exploration of the "continuous cochain" theory [**12**]. These were based on the usual cochain definition of group cohomology, except that now cochains were smooth / continuous. The smooth cochain theory was somewhat specialized, and the continuous cochain theory did not give all the properties one wanted for a cohomology theory of groups. For example, the theory does not give a long exact sequence on cohomology for many short exact sequences of topological G-modules.

Calvin Moore [**21**] defined a somewhat less restricted theory $H^n_m(G, A)$ than van Est's, the measurable cochain theory, and showed it has all the properties listed above. The main issue with this theory is that it is difficult to compute the cohomology groups; a slightly less concerning issue is that G is restricted to be a Hausdorff locally compact second countable group and A must be a Polish group (i.e. a second countable Hausdorff complete metric group).

Graeme Segal [**29**] constructed a classifying space for an arbitrary category and used it to construct a CW complex BG which acts as the classifying space for any topological group. One can then compute the sheaf cohomology $H^n(BG, A)$ for the locally constant sheaf A, but this cannot account for any topology on A.

David Wigner, in [**35**] and his thesis [**36**], defined several new cohomology theories by using Yoneda Ext's, which naturally satisfy the universality property mentioned above. He also constructed another theory $H^n_{ss}(G, A)$ using a semisimplicial complex of sheaves. He showed $H^n_{ss}(G, A) = H^n(BG, A)$ when A is discrete and also showed that $H^n_{ss}(G, A)$ coincides with $H^n_m(G, A)$ under some special conditions. My starting point in exploring the theory of topological group cohomology was to give complete proofs of some of the claims Wigner made in [**35**]; these are presented in Chapters 3 and 4.

G. J. Mitchison and Segal also developed another cohomology theory. By restricting to locally contractible compactly generated Hausdorff G-modules A, they were able to obtain contractible resolutions for all objects, and thus their theory satisfied the universality condition mentioned above. Various other theories have since been developed, and many are comparable in this setting [**34**].

As of yet, Grothendieck topologies had not entered into the picture. Stephen Lichtenbaum [**17**] gave the first interpretation of one of the established cohomology theories, Wigner's $H^n_{ss}(G, A)$, in terms of a Grothendieck topology and used it to develop a "Weil-étale topology" for number fields and formulate a deep conjecture about special values of zeta functions for a number field. Inspired by his work, Arati Khedekar and C. S. Rajan [**15**] defined the "locally continuous measurable cochain theory" in an attempt at connecting topological group cohomology with the Langlands program. M. Flach also used Grothendieck topologies to define a cohomology theory for topological groups which was applied to the Weil-étale topology [**8**].

With so many cohomology theories out there, one of the most important questions now is, when do they give the same results? One of the main ideas in this thesis is that Grothendieck topologies may play an important role in the cohomology theory for topological groups because they can be used to compare different cohomology theories. Another advantage is their applicability, such as in [**17**]. A third is that they give long exact sequences on cohomology for a specific class K of short exact sequences, and it is very easy to give an explicit description of K from the definition of the (coverings of the) Grothendieck topology (see Section 2.3).

In Chapter 2 we define several Grothendieck topologies whose cohomology is the same as some of the well-established theories, namely the continuous cochain theory, a natural generalization of the measurable cochain theory and the locally continuous measurable cochain theory, and the locally continuous cochain theory [**34**]. We also define some new cohomology theories which seem natural and construct maps comparing various theories by using morphisms of Grothendieck topologies.

Notation

$\mathcal{S}(T)$ is the category of sheaves of abelian groups on the Grothendieck topology T

$\mathcal{S}'(T)$ is the category of sheaves of sets on the topology T

$\mathrm{Cat}(T)$ refers to the category underlying the topology T

G**-mod** is the category of G-modules

G**-set** is the category of G-sets

T_G is the canonical topology on G**-set**

$\mathcal{C}_G$ is the category of G-spaces for a topological group G

$\mathcal{M}_G$ is the category of topological G-modules

$\tilde{A}$ is the sheaf $\mathrm{Hom}(-, A)$ on any topology

$\varnothing$ refers both to the empty set and any map out of one

pt is a set with one element

e is used to denote the identity in a group

1 is used to denote the group with one element

When the special element of a certain set is obvious, it will sometimes be omitted. For example, the group G may denote the pointed set (G, e).

A neighborhood of a point x in a topological space means a set which contains an open set U with $x \in U$.

CHAPTER 1

Background

This chapter is devoted to basic results and definitions we will use later. In Section 1.1 we go over the basic definitions and constructions in the theory of Grothendieck topologies, mostly following [**31**]. Then we state some technical lemmas about maps of sheaves (Section 1.1.2) and comparisons of Grothendieck topologies (Section 1.1.3).

In Section 1.2 we go over Yoneda's definition of $\mathrm{Ext}^n(A,B)$ in the setting of quasi-abelian S-categories in detail. The Yoneda pullback is defined in Section 1.2.3 and is very important in the context of Grothendieck topologies (Section 2.3). In Section 1.2.5 we prove a technical result which will allow us to compare the Ext groups in different categories; we will use this result in Chapter 4. We show that the Ext groups satisfy a universality condition with respect to a certain class of short exact sequences in Section 1.2.6. This shows the importance of cohomology theories defined using the Yoneda Ext's.

In Section 1.3 we go over some basic properties of topological groups G, G-spaces, and topological G-modules. Of particular importance is Section 1.3.3.2, where we set the notation for the categories of G-modules we will use and prove that these categories are quasi-abelian.

1.1. Grothendieck Topologies

1.1.1. Basics.

Definition 1.1.1. *A* **Grothendieck topology**[1] *T is a category, denoted* $\mathrm{Cat}(T)$, *together with a collection of coverings $\{X_i \to X\}_{i\in I}$ for every object X in* $\mathrm{Cat}(T)$, *satisfying*

[1] This was actually called a pretopology by Grothendieck.

the conditions below (so every object has associated with it a collection of such families of maps $\{X_i \to X\}$).

(1) *If $f : X \to Y$ is an isomorphism, then $\{f : X \to Y\}$ is a covering.*

(2) *If $\{X_i \to X\}$ is a covering and $f : Y \to X$ is any morphism, then all fibered products $X_i \times_X Y$ exist and $\{X_i \times_X Y \to Y\}$ is a covering (this property is referred to as stability under base change).*

(3) *If $\{X_i \to X\}_{i \in I}$ is a covering and for each i, $\{X_{i,j} \to X_i\}_{j \in J_i}$ is a covering, then $\{X_{i,j} \to X\}_{i \in I, j \in J_i}$ is a covering.*

Presheaves and Sheaves. A **presheaf** of sets (resp. abelian groups, etc.) on T is a contravariant functor from $C = \mathrm{Cat}(T)$ to the category of sets (resp. abelian groups, etc.). For a presheaf F, elements of $F(X)$ are called **sections**. For a map $f : X \to Y$ and a section $s \in F(Y)$, we say $F(f)s$ is the **restriction** of s to X, denoted $s|X$. A **sheaf** (of sets, abelian groups, etc.) is a presheaf satisfying the property that for every covering $\{X_i \to X\}_{i \in I}$ in T the following diagram is exact:

$$F(X) \to \prod_{i \in I} F(X_i) \rightrightarrows \prod_{i,j \in I} F(X_i \times_X X_j)$$

This means for every collection of elements $s_i \in F(X_i)$ such that the restriction $s_i|X_i \times_X X_j$ of s_i to $X_i \times_X X_j$ is the same as the restriction of s_j for all $i, j \in I$, there is a unique element $s \in F(X)$ with $s_i = s|X_i$ for all $i \in I$.

A morphism of presheaves (or sheaves) is just a natural transformation. A presheaf F is **representable** by the object X in C if $F \cong \mathrm{Hom}(-, X)$. If all representable presheaves are sheaves, then the topology T is called **subcanonical**. There is a unique largest subcanonical topology T on a category C, and it is called the **canonical** topology. A **Grothendieck topos** is a category which is equivalent to the category of sheaves on some topology.

Cohomology. It turns out that the category $\mathcal{S}(T)$ of sheaves of abelian groups on T is always an abelian category that has enough injectives, so that we can take injective resolutions of sheaves and define derived functors. For an object X in C and presheaf F on T, we define $\Gamma(X, F) = F(X)$. The functor $\Gamma(X, -)$ from $\mathcal{S}(T)$ to **Ab**

has right-derived functors $H^n(X, -)$, and we define the **cohomology** of T as $H^n(X, F)$ in this way. If it is necessary to emphasize the topology, we write $H^n(T, X, F)$.

Main Example. Let G be a group and G**-set** the category of G-sets and G-maps. The usual topology T_G on this category is defined by setting the covers to be families $\{X_i \xrightarrow{f_i} X\}$ such that $X = \bigcup f_i(X_i)$. It turns out that this is the canonical topology and the category of sheaves of sets on T_G is equivalent to G**-set** ; more precisely, all sheaves are representable. This means G**-set** is a Grothendieck topos. The category of sheaves of abelian groups on T_G is equivalent to G**-mod** , the category of G-modules; G-modules are precisely the abelian group objects in G**-set** . For a G-module A, we denote by $\tilde{A}$ the sheaf $\mathrm{Hom}(-, A)$. The cohomology $H^n(T_G, pt, \tilde{A})$ is precisely the usual group cohomology $H^n(G, A)$.

Sheafification. There is a left adjoint to the natural inclusion of the category of sheaves on a topology T into the category of presheaves. Given a presheaf P, we define

$$P^\dagger(U) = \varinjlim_{\{U_i \to U\}} \ker\left(\prod P(U_i) \rightrightarrows \prod P(U_i \times_U U_j)\right)$$

where the limit is taken over all coverings $\{U_i \to U\}$ and a map from one covering $\{U_i \xrightarrow{f_i} U\}_{i \in I}$ to another $\{V_j \xrightarrow{g_j} U\}_{j \in J}$ consists of a map $\phi : I \to J$ of indices and maps $h_i : U_i \to V_{\phi(i)}$ such that $g_{\phi(i)} \circ h_i = f_i$. A map of coverings is called a **refinement**, and if there is a map from a covering $\mathcal{U}$ to another covering $\mathcal{V}$, we say $\mathcal{U}$ **refines** $\mathcal{V}$. Then the left adjoint desired is given by $P^\# = P^{\dagger\dagger}$. $P^\#$ is called the **sheafification** of, or the **sheaf associated** to P.

Čech Cohomology. The Čech cohomology $H^n(\{U_i \to U\}_{i \in I}, F)$ for a presheaf F and a covering $\{U_i \to U\}$ is defined to be the n-th cohomology of the complex $(C^n(\{U_i \to U\}_{i \in I}, F), \delta_n)_{n \geq 0}$ where

$$C^n(\{U_i \to U\}_{i \in I}, F) = \prod_{i_0, \dots, i_n \in I^{n+1}} F(U_{i_0} \times_U \cdots \times_U U_{i_n})$$

and $\delta_n : C^n(\{U_i \to U\}_{i \in I}, F) \to C^{n+1}(\{U_i \to U\}_{i \in I}, F)$ is given by

$$\delta_n(s)_{i_0, \dots, i_{n+1}} = \sum_{k=0}^{n+1} (-1)^k (s_{i_0, \dots, \hat{i_k}, \dots, i_{n+1}} \mid U_{i_0} \times_U \cdots \times_U U_{i_{n+1}})$$

The Čech cohomology $H^n(U, F)$ is the direct limit of $H^n(\{U_i \to U\}_{i\in I}, F)$ over all the coverings $\{U_i \to U\}_{i\in I}$ of U. Note that the Čech cohomology $H^n(\{U_i \to U\}_{i\in I}, F)$ of a covering does not depend on the topology T, but the Čech cohomology $H^n(U, F)$ does.

Morphism of Topologies. If we have two topologies T and T', a **morphism of topologies** $g : T \to T'$ is a functor from $\mathrm{Cat}(T)$ to $\mathrm{Cat}(T')$ which preserves the final object and fibered products[2], and takes coverings to coverings. Associated to g are two functors on categories of sheaves. The first, the **direct image** $g_* : \mathcal{S}(T') \to \mathcal{S}(T)$ is defined by $g_*F(X) = F(gX)$. The second, the **inverse image** $g^* : \mathcal{S}(T) \to \mathcal{S}(T')$ is the left adjoint of g_*; it is more complicated to describe. g^*F is the sheaf associated to the presheaf g^pF given by

$$g^pF(Y) = \varinjlim_{Y\to gX} F(X)$$

where Y is an object of T' and the limit is taken over maps $Y \to gX$, where a morphism from one such map $Y \xrightarrow{f_1} gX_1$ to another map $Y \xrightarrow{f_2} gX_2$ is a map $h : X_1 \to X_2$ with $g(h) \circ f_1 = f_2$.

1.1.2. Fundamental Lemmas.

Lemma 1.1.2. [**31**, Theorem I.3.7.6] *Let $g : T \to T'$ be a morphism of topologies. Then there exists a Leray spectral sequence $E_2^{pq} = H^p(T, X, R^q g_* F) \Rightarrow H^{p+q}(T', g(X), F)$ for any X in* $\mathrm{Cat}(T)$*, canonical in the sheaf F.*

Lemma 1.1.3. [**31**, Theorem I.3.9.2] *Let $g : T \to T'$ be a morphism of topologies. Then for any sheaf T on T', $R^n g_* F$ is the sheaf on T associated to the presheaf $X \mapsto H^n(T', g(X), F)$.*

[2]Actually, for the lemmas below, it is enough that for any covering $\{X_i \to X\}$ and any morphism $Y \to X$ we have $g(X_i \times_X Y) \cong g(X_i) \times_{g(X)} g(Y)$, but we do not lose anything by simply assuming g preserves all fibered products.

We recall a definition from Milne's notes on étale cohomology[3] and prove some basic lemmas we will later use.

Definition 1.1.4. *A map* $F \xrightarrow{\alpha} F'$ *of presheaves is* **locally surjective** *if for every section* $s \in F'(U)$ *there is a covering* $\{U_i \to U\}_{i \in I}$ *such that* $s|U_i$ *is in the image of* α *for all* $i \in I$.

Lemma 1.1.5. *A map* $\alpha : F \to F'$ *of sheaves is an epimorphism in the category of sheaves if and only if it is locally surjective.*

Proof. Suppose α is an epimorphism, and let $s \in F'(U)$. Let P be the presheaf $\operatorname{coker} \alpha : U \mapsto F'(U)/\operatorname{Im}(F(U))$. Then the sheaf $\operatorname{coker} \alpha$ is the sheafification $P^\#$ of P [**31**, p. 51], and $P^\# = 0$. Since the map $P^\dagger \to P^\#$ is injective [**31**, Proposition I.3.1.3], this implies $P^\dagger = 0$. Thus, by the definition of $P^\dagger(U)$ as a direct limit, the section $\bar{s} \in P(U) = F'(U)/\operatorname{Im}(F(U))$ is locally equal to 0, that is, there is a covering $\{U_i \to U\}_{i \in I}$ such that $\bar{s}|U_i = 0$ for all $i \in I$. But that means $s|U_i \in \operatorname{Im}(F(U_i))$, so α is locally surjective.

Conversely, suppose α is locally surjective. To show α is an epimorphism, suppose we have a map $\beta : F' \to G$ of sheaves, such that $\beta \circ \alpha = 0$; we need to show $\beta = 0$, so let $s \in F'(U)$. There exist a covering $\{U_i \to U\}_{i \in I}$ and sections $s_i \in F(U_i)$ such that $s|U_i = \alpha(s_i)$ for all $i \in I$. Thus $\beta(s)|U_i = \beta(s|U_i) = \beta(\alpha(s_i)) = 0$, and so, by the sheaf axiom for G, $\beta(s) = 0$, as desired. □

The following lemma is a slight strengthening of [**31**, Proposition I.3.7.1].

Lemma 1.1.6. *Suppose there is a morphism of topologies* $g : T \to T'$ *such that for every object* X *of* T *and every cover* $\{Y_i \to g(X)\}$ *in* T' *there is a cover* $\{f_j : X_j \to X\}$ *in* T *such that* $\{g(f_j) : g(X_j) \to g(X)\}$ *refines* $\{Y_i \to g(X)\}$. *Then* g_* *is exact, hence for any sheaf* F *on* T' *and any object* X *of* T *we have* $H^n(T, X, g_*F) \cong H^n(T', g(X), F)$.

Proof. One can use an argument similar to that used in [**1**, Corollary 2.4.7], explicated in [**31**, Lemma I.3.8.1], but to keep the text self-contained we provide a

[3] http://www.jmilne.org/math/CourseNotes/LEC.pdf

different argument based on Lemma 1.1.5: g_* is always left exact, so we just have to show that if $\alpha : F \to F'$ is an epimorphism of sheaves then $g_*\alpha : g_*F \to g_*F'$ is locally surjective. So let $s \in g_*F'(X) = F'(g(X))$. There is a cover $\{Y_i \to g(X)\}_{i\in I}$ in T' such that for each i, $s|Y_i = \alpha(s_i)$ for some $s_i \in F(Y_i)$. By the hypothesis of the lemma, there is a cover $\{f_j : X_j \to X\}_{j\in J}$ such that $\{g(f_j) : g(X_j) \to g(X)\}_{j\in J}$ refines $\{Y_i \to g(X)\}_{i\in I}$. This means there is a map $\phi : J \to I$ such that $g(X_j) \to g(X)$ factors through $Y_{\phi(j)} \to g(X)$ for each $j \in J$. Therefore, $s|g(X_j) = (s|Y_{\phi(j)})|g(X_j) = \alpha(s_{\phi(j)})|g(X_j) = \alpha(s_{\phi(j)}|g(X_j)) \in F'(g(X_j)) = g_*F'(X_j)$, i.e. $s|X_j \in \mathrm{Im}(g_*\alpha)$ for all $j \in J$, which says that $g_*\alpha$ is locally surjective, as desired.

Now consider the Leray spectral sequence

$$E_2^{pq} = H^p(T, X, R^q g_* F) \Rightarrow H^{p+q}(T', g(X), F).$$

Since g_* is exact, $R^q g_* F = 0$ for $q > 0$, so the spectral sequence gives the isomorphism stated in the conclusion of the lemma. □

Lemma 1.1.7. [**31**, p. 59] *If* $\{U_i \to U\}_{i\in I}$ *is a covering in a Grothendieck topology and* F *is an abelian sheaf such that* $H^q(U_{i_0} \times_U \cdots \times_U U_{i_n}, F) = 0$ *for all* $q > 0$, $n \geq 0$, *and all tuples* $(i_0, \ldots, i_n) \in I^{n+1}$ *then there are canonical isomorphisms* $H^p(\{U_i \to U\}, F) \xrightarrow{\sim} H^p(U, F)$ *for all* p.

1.1.3. Comparison of Grothendieck Topologies. We introduce the terminology in Vistoli's *Grothendieck topologies, fibered categories and descent theory* [**33**] used to compare two different topologies on the same category; this will be used in Chapter 2, particularly in Section 2.2. If there are two topologies T_1, T_2 on a category, we say T_2 is **finer** than T_1, or T_1 is **coarser** than T_2, written $T_1 \prec T_2$, if every covering in T_1 has a refinement in T_2. $T_1 \prec T_2$ if and only if every sheaf of sets for T_2 is a sheaf for T_1 (see [**33**] and Vistoli's post on Mathoverflow[4]).

If $T_1 \prec T_2$ and $T_2 \prec T_1$ then we say the two topologies are **equivalent**; this implies they have the same sheaves and the same cohomology (by Lemma 1.1.6).

[4]http://mathoverflow.net/questions/128564/can-inequivalent-topologies-have-same-sheaves-cohomology

If $T_1 \prec T_2$ but not $T_2 \prec T_1$, we will write $T_1 \precneqq T_2$; this implies that there are strictly more sheaves for T_1 than for T_2.

A topology T is called **saturated** if whenever a family $\mathcal{U} = \{U_i \to U\}$ has a refinement by a covering in T, $\mathcal{U}$ itself is a covering in T. Any topology T is equivalent to its **saturation**: the topology whose coverings are those which have a refinement in T. It is easy to check that the saturation of a topology is indeed a topology.

1.2. Quasi-Abelian Categories

1.2.1. S-Categories. Two standard references for quasi-abelian categories are Nobuo Yoneda's paper [**39**], where he works in quasi-abelian categories and constructs higher Ext groups in terms of exact sequences, proving that a short exact sequence gives a long exact sequence of Ext's, and [**28**], which uses Grothendieck's definition of quasi-abelian categories and details the construction of derived categories using quasi-abelian categories. A very closely related paper on exact categories, where the axioms are essentially the same, is [**5**].

A morphism $f : A \to B$ in an additive category is **proper** if the natural map from $\operatorname{coker}(\ker f) \to \ker(\operatorname{coker} f)$ is an isomorphism and, in particular, these kernels and cokernels exist. Note that, if all kernels and cokernels exist, then a morphism is a proper monomorphism $\iff$ it is the kernel of some morphism $\iff$ it is the kernel of its own cokernel. Dually, a morphism is a proper epimorphism $\iff$ it is the cokernel of some morphism $\iff$ it is the cokernel of its own kernel.

An S**-category** is an additive category C together with a class S of morphisms of C such that:

(S1) All isomorphisms are in S, and all maps in S are proper.[5]

[5]Actually, Yoneda has a small typo and writes $\mathbf{P}\mathscr{A} \supset S \cap \mathbf{E}\mathscr{A}$, where $\mathscr{A}$ is our C, $\mathbf{P}\mathscr{A}$ is the class of proper morphisms in $\mathscr{A}$, and $\mathbf{E}\mathscr{A}$ is the class of "equivalence maps," or isomorphisms in modern language. He later says that the smallest possible S is the set of **direct maps**, i.e. proper maps $\phi : A \to B$ such that there is a map $\psi : B \to A$ with $\phi\psi\phi = \phi$, and the largest possible S is $\mathbf{P}\mathscr{A}$, so it is clear that he meant $\mathbf{P}\mathscr{A} \supset S \supset \mathbf{E}\mathscr{A}$.

(S2) For any two morphisms $f : A \to B, g : C \to D$ in S, the morphism $f \oplus g : A \oplus C \to B \oplus D$ is in S.

(S3) If $\phi \in S$ then all kernels and cokernels of ϕ are in S.

(S4) Any $f \in S$ can be written $f = m \circ e$, where $m, e \in S, \ker m = 0, \operatorname{coker} e = 0$, and any such composition $m \circ e$ is in S.

The class $P(\mathcal{C})$ of all proper morphisms satisfies all these properties and so is the largest possible class S for $\mathcal{C}$.

In working with Grothendieck topologies as we do here, one wants a list of axioms that ensures a class of epimorphisms gives an S-category; we present one such list here. Some of these axioms become redundant if one includes the Axioms (Q) and (Q*) (see Section 1.2.2 below) to ensure we have a quasi-abelian S-category.

Proposition 1.2.1. *Given a class E of epimorphisms in $\mathcal{C}$, there is a unique class S such that $(\mathcal{C}, S)$ is an S-category and E is the subclass of epimorphisms in S, provided E satisfies the following conditions:*

(1) all identity maps are in E,

(2) all maps $X \to 0$ for $X \in \operatorname{Ob} \mathcal{C}$ are in E,

(3) all maps in E are proper,

(4) if $e, e' \in E$ then $e \oplus e' \in E$,

(5) for any $e : A \to I$ in E and any isomorphism $\phi : I \to B$, the composition $\phi \circ e$ is in E.

Proof. To construct S, simply let M be the class of all kernels of maps in E (two different kernels of the same map are not identified, even though they are uniquely isomorphic) and define S as $M \circ E$, the class of compositions $m \circ e$ such that $m \in M, e \in E$. For any $X \in \operatorname{Ob} \mathcal{C}$, we have $\mathrm{id}_X \cong \ker(X \to 0) \in M$, so $E \subseteq S$, and $M \subseteq S$ by (1). Any isomorphism $\phi : A \xrightarrow{\sim} B$ is a kernel of $B \to 0$, so $\phi \in M$ and isomorphisms are in S. All maps in M are proper, since they are kernels of their own cokernels by (3), so to see that S satisfies (S1) we just have to show that any composition $A \xrightarrow{e} I \xrightarrow{m} B$ with m a kernel and e a cokernel is proper. Indeed, $\ker(m \circ e) = \ker(e)$, and since e is a cokernel, e is the cokernel of its own kernel, so

$\operatorname{coker}(\ker(m \circ e)) \cong e$. Dually, $\operatorname{coker}(m \circ e) = \operatorname{coker}(m)$ and $\ker(\operatorname{coker}(m \circ e)) \cong m$, so $m \circ e$ is proper.

To show that S satisfies (S2), suppose we have two compositions $A \xrightarrow{e} I \xrightarrow{m} B$ and $A' \xrightarrow{e'} I' \xrightarrow{m'} B'$ with $m, m' \in M, e, e' \in E$. Then $e \oplus e' \in E$, and if $m = \ker(f), m' = \ker(f')$ for some $f, f' \in E$, then $f \oplus f' \in E$ so $m \oplus m' = \ker(f \oplus f') \in M$ ($\oplus$ is exact), hence the map $(m \circ e) \oplus (m' \circ e')$, which can be decomposed as $A \oplus A' \xrightarrow{e \oplus e'} I \oplus I' \xrightarrow{m \oplus m'} B \oplus B'$, is in S. Next, keeping the same notation, $\ker(m \circ e) = \ker(e) \in M$, so $\ker(m \circ e) = \ker(e) \circ \mathrm{id}_A \in S$, and $\operatorname{coker}(m \circ e) = \operatorname{coker}(m) \cong f \in E$ so $\operatorname{coker}(m \circ e) \in S$ by (5), hence S satisfies (S3). Now if $m \circ e$ is epic then $\operatorname{coker}(m \circ e) = \operatorname{coker}(m) = 0$, so m is an isomorphism and $m \circ e \in E$ by (5), so E is the class of epimorphisms in S. Similarly, if $m \circ e$ is monic, then e is an isomorphism, so $A \xrightarrow{e} I \xrightarrow{m} B$ is a kernel of f, hence $m \circ e \in M$, which shows that M is the class of monomorphisms in S. Finally, S was defined to be $M \circ E$, so it satisfies (S4).

The uniqueness of S can be verified by first noting that the kernels of $e \in E$ must be in S and all compositions $m \circ e$ with $m \in M, e \in E$ must be in S, so any other class S' with $E \subseteq S'$ must satisfy $S \subseteq S'$. Now if $g : A \to B$ is a map in S' then g must factor as $A \xrightarrow{e} I \xrightarrow{m} B$ with e epic and m monic, $e, m \in S'$, and if all the epimorphisms in S' are to be in E, we must have $e \in E$ and $\operatorname{coker}(m) \in E$, hence $m \in M$ and $g = m \circ e \in S$. □

Definition 1.2.2 (Yoneda). *A sequence $A \xrightarrow{a} B \xrightarrow{b} C$ is* **exact** *if the maps a and b are proper, $b \circ a = 0$, and the natural map* $\operatorname{Im}(a) \to \ker(b)$ *is an isomorphism.*

Corollary 1.2.3. *Given a class K of short exact sequences $0 \to A \to B \xrightarrow{\tau} C \to 0$ such that the class E of epimorphisms τ satisfies the conditions of Proposition 1.2.1, there is a unique class S such that $(\mathcal{C}, S)$ is an S-category and the class of short exact sequences in S is K.*[6]

[6]Technically, we should require that if $A \cong A'$ and $0 \to A \to B \xrightarrow{\tau} C \to 0$ is in K, then $0 \to A' \to B \xrightarrow{\tau} C \to 0$ is also in K.

Proof. Construct the class S from E as in the proof of the proposition. An exact sequence $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ is in $S \iff \alpha, \beta \in S \iff \beta \in S$ since all kernels of $\beta \in E$ are in S. Thus K is exactly the class of short exact sequences in S. Conversely, if S' is another class making (C, S) an S-category such that K is the class of short exact sequences in S', then the class of epimorphisms $B \xrightarrow{\tau} C$ in S' is precisely E (since any epimorphism in S' must be the cokernel of its own kernel, hence gives a short exact sequence) so, by the uniqueness of S in Proposition 1.2.1, we must have $S = S'$. □

Lemma 1.2.4. *In the short exact sequence* $0 \to B \xrightarrow{\beta} E \xrightarrow{\alpha} A \to 0$, *$\alpha$ has a section, i.e. a morphism* $s : A \to E$ *with* $\alpha s = \mathrm{id}_A \iff$ *there is a morphism* $t : E \to B$ *with* $\beta t = \mathrm{id}_B$.

Proof. $\alpha(\mathrm{id}_E - s\alpha) = 0$ implies that there is a unique morphism $t : E \to B$ with $\beta t = \mathrm{id}_E - s\alpha$. Then $\beta t \beta = \beta - s\alpha\beta = \beta \Rightarrow t\beta = \mathrm{id}_B$. Dually, if t exists, then so does s. □

1.2.2. Quasi-Abelian S-Categories.

Definition 1.2.5. *An S-category is* **quasi-abelian** *if it satisfies the following four conditions. An additive category C is* **quasi-abelian** *if $(C, P(C))$ satisfies (Q2) and (Q2*) below (Yoneda* [**39**, p. 522] *proves that a quasi-abelian category C automatically satisfies (Q0) and (Q0*) below).*

- (Q0) A composition of epimorphisms in S is an epimorphism in S.
- (Q0*) A composition of monomorphisms in S is a monomorphism in S.
- (Q2) Any exact sequence $0 \to B \to E \to A \to 0$ in S and any map $A' \to A$ can be embedded in a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & B & \to & E' & \to & A' & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & B & \to & E & \to & A & \to & 0 \end{array}$$

with both rows exact and in S.

(Q2*) Any exact sequence $0 \to B \to E \to A \to 0$ in S and any map $B \to B'$ can be embedded in a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & B & \to & E & \to & A & \to & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & B' & \to & E' & \to & A & \to & 0 \end{array}$$

with both rows exact and in S.

Yoneda proves that these four axioms together are equivalent to the following two:

(Q) Any exact sequence $0 \to B \to E \to A \to 0$ in S and any monomorphism $A' \to A$ in S can be embedded in a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & B & \to & E' & \to & A' & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & B & \to & E & \to & A & \to & 0 \end{array}$$

with both rows exact and all maps in S (note the map $E' \to E$ is a priori a monomorphism).

(Q*) Any exact sequence $0 \to B \to E \to A \to 0$ in S and any epimorphism $B \to B'$ can be embedded in a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & B & \to & E & \to & A & \to & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & B' & \to & E' & \to & A & \to & 0 \end{array}$$

with both rows exact and all maps in S.

Wigner **[35]** defines a quasi-abelian S-category using a class of short exact sequences

(1) $$0 \to A \xrightarrow{i} B \xrightarrow{d} C \to 0$$

instead of a class of morphisms. It can easily be seen that (Q) is equivalent to Wigner's (Q) **[35]**, which says that any exact sequence $0 \to B \to E \to A \to 0$ in S

and any monomorphism $A' \to A$ in S can be embedded in a commutative diagram

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \to & B & \to & E' & \to & A' & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & B & \to & E & \to & A & \to & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & C & = & C & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array}$$

with both rows and columns exact and all maps in S [**35**]. Of course, dually, (Q*) is equivalent to Wigner's (Q*), which is the dual of his (Q).

Technically, if one is to use short exact sequences, one needs some further assumptions such as those in Proposition 1.2.1, those made by Quillen [**27**], or the shorter list made by Keller [**14**, Appendix A]:

Theorem 1.2.6 (Keller). *Let K be a class of short exact sequences (1) as defined by Yoneda in the additive category $\mathcal{C}$, E the class of epimorphisms β in K, and M the class of monomorphisms α in K. If the following assumptions hold, then K gives rise to a quasi-abelian S-category:*

(K0) K is closed under isomorphism,

(K1) $\mathrm{id}_0 \in E$,

(K2) E is closed under composition,

(K3) E is closed under pullback by any morphism,

(K3) M is closed under pushout by any morphism.*

Proof. We show that the class E of epimorphisms satisfies properties (1)-(5) of Proposition 1.2.1. (1) For any object A in $\mathcal{C}$, the pullback of id_0 by $A \to 0$ is id_A. (2) For any object A in $\mathcal{C}$, $\mathrm{id}_A \in E \Rightarrow (0 \to A) \in M$. Take the pushout of $0 \to A$ by $0 \to A$ to get $A \to A \oplus A$ in M, so the short exact sequence $0 \to A \to A \oplus A \to A \to 0$

is in K. Taking the pullback of $A \oplus A \to A$ by $0 \to A$, we see the map $A \to 0$ is in E. (3) By definition of Yoneda's exact sequences, all maps in E are proper. (4) Given epimorphisms $e : B \to C, e' : B' \to C'$ in E, pull back e by the projection $C \oplus B' \to C$ to get $e \oplus \mathrm{id} : B \oplus B' \to C \oplus B'$ in E; similarly, $\mathrm{id} \oplus e' : C \oplus B' \to C \oplus C'$ is in E, hence so is $e \oplus e' = (e \oplus \mathrm{id}) \circ (\mathrm{id} \oplus e')$. (5) For any $e : B \to C$ in E and any isomorphism $\phi : C \to C'$, the short exact sequence containing e is isomorphic to the one containing $\phi \circ e$. Proposition 1.2.1 then says we have an S-category. Keller[**14**] proves these axioms imply (K2*): M is closed under composition, so the axioms (Q0), (Q0*), (Q2), (Q2*) are all satisfied and (C, S) is a quasi-abelian S-category. □

Theorem 1.2.7. *(Q2) and (Q2*) are equivalent, respectively, to Grothendieck's axioms (QA) and (QA*):*

(QA) The pullback of an epimorphism in S by any map exists and is (an epimorphism) in S.

(QA) The pushout of a monomorphism in S by any map exists and is (a monomorphism) in S.*

Proof. We show (QA) $\iff$ (Q2); the proof that (QA*) $\iff$ (Q2*) is dual; first suppose (QA) holds. Given the exact sequence $0 \to B \xrightarrow{a'} E \xrightarrow{f'} A \to 0$ and the monomorphism $A \to A'$ in S, let $E' = E \times_A A'$. By the universal property of the fibered product, there is a unique map $B \to E'$ such that the composition $B \to E' \to A'$ is zero and the left square in the diagram for (Q2) is commutative. Now the projection $E' \to A'$ is (by (QA)) an epimorphism in S, so it is proper, which means it is the cokernel of its own kernel, so we just have to show $B \to E'$ is the kernel of $E' \to A'$, which is elementary and is left as an exercise.

Conversely, suppose (Q2) holds. The following argument is due to Theo Bühler[7]. Let $E \xrightarrow{e} A$ be an epimorphism in S and $f : A' \to A$ be any map. Let $(B \to E)$ be the kernel of e. It is well-known that f factors as $A' \xrightarrow{f'} A \oplus A' \xrightarrow{\pi_1} A$, where f' is given by $f'(a) = (f(a), a)$ (translated into the language of additive categories) and π_1 is

[7] Private communication.

the projection onto the first coordinate. If we take the direct sum of the sequences $0 \to B \to E \to A \to 0$ and $0 \to 0 \to A' \xrightarrow{\mathrm{id}} A' \to 0$, and then apply (Q2) to the resulting sequence we get the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \to & B & \to & E' & \xrightarrow{e'} & A' & \to & 0 \\
 & & \| & & \downarrow^{g} & & \downarrow^{f'} & & \\
0 & \to & B & \to & E \oplus A' & \xrightarrow{e \oplus \mathrm{id}_{A'}} & A \oplus A' & \to & 0 \\
 & & \| & & \downarrow^{\pi_1} & & \downarrow^{\pi_1} & & \\
0 & \to & B & \to & E & \xrightarrow{e} & A & \to & 0
\end{array}$$

The morphism $E \oplus A' \to A \oplus A'$ is the direct sum of e and $\mathrm{id}_{A'}$, so it is in S. By (Q2), the map $E' \to A'$ is in S. Now it is easy to see that the bottom-right square and the top-right square are pullback squares, so $E' \cong E \times_A A'$ and the pullback of e by f is an epimorphism in S, as desired. □

1.2.3. Yoneda Pullback.

Definition 1.2.8. *Let (C', S') be an S-category, C an additive category, and $f : C \to C'$ an additive functor. We define* **Yoneda's pullback** *class $f^{\#}S'$ on C as the set of proper maps $\phi : A \to B$ such that f takes the exact sequences $0 \to \ker \phi \to A \to \mathrm{Im}\,\phi \to 0$ and $0 \to \mathrm{Im}\,\phi \to B \to \mathrm{coker}\,\phi \to 0$ to exact sequences.*

The class $f^{\#}S'$ is the largest class S on C under which (C, S) is an S-category such that for all $\phi \in S$ we have $f(\phi) \in S'$. Note that if (C, S_i) is an S-category for all i in some index set I then $(C, \bigcap S_i)$ is an S-category, so in particular if (C, S) is an S-category then so is $(C, S \cap f^{\#}S')$. Yoneda proved the following theorem [**39**, p. 531]:

Theorem 1.2.9. *If $f : C \to C'$ is* **half-exact** *(also known as exact in the middle), i.e. takes an exact sequence $0 \to A \to B \to C \to 0$ to an exact sequence $fA \to fB \to fC$, and $(C', S'), (C, S)$ are quasi-abelian S-categories then $(C, S \cap f^{\#}S')$ is a quasi-abelian S-category. In particular, if C is a quasi-abelian category, (C', S') is a quasi-abelian S-category, and f is half-exact, then $(C, f^{\#}S')$ is a quasi-abelian S-category.*

In [**35**], Wigner talks about a class S in which the short exact sequences $0 \to A \to B \xrightarrow{\tau} C \to 0$ satisfy the property that τ has a continuous (not necessarily G-equivariant) section $\sigma : U \to B$ with $\tau \circ \sigma = \mathrm{id}_C$. This class is also used in [**12**]. The class E of all such proper epimorphisms τ satisfies the five conditions of Proposition 1.2.1, so there is indeed a unique class S in which E is the subclass of epimorphisms and K_E is the class of short exact sequences in S. Another explicit description of this class S when C is the category $\mathcal{M}_G$ of topological G-modules is $y^{\#}S'$, where $S' = \mathrm{Mor}(\mathcal{S}(T^c_G))$ (see Section 2.3) is the class of all morphisms in the category of sheaves on the global-section topology T^c_G defined in Section 2.1.2 and $y : \mathcal{M}_G \to \mathcal{S}(T^c_G) : A \mapsto \tilde{A}$ is the Yoneda embedding. This shows that $(\mathcal{M}_G, S)$ is a quasi-abelian S-category by Theorem 1.2.9.

The class of short exact sequences $0 \to A \to B \xrightarrow{\tau} C \to 0$ of Polish topological G-modules where τ has a local section around $0 \in C$ is used extensively in Khedekar and Rajan's locally continuous cochain theory [**15**]. More precisely, the corresponding epimorphisms $\tau : B \to C$ satisfy the following condition: there is a neighborhood U of $0 \in C$ and a continuous map $\sigma : C \to B$ such that $\tau \circ \sigma = \mathrm{id}_U$. It is easy to check that this class of epimorphisms E satisfies the conditions of Proposition 1.2.1, hence gives a class S which makes $(\mathcal{M}_G, S)$ an S-category. But another description of this class is that the epimorphisms $\tau : B \to C$ in S have local sections everywhere, that is, for all $x \in C$ there is a neighborhood U of x in C and a continuous map $\sigma : U \to B$ with $\tau \circ \sigma = \mathrm{id}_U$. This is because if we have a local section σ on a neighborhood U of $0 \in C$, then we can translate it to a local section σ_x on $U + x$ defined by $\sigma_x(y) = \sigma(y - x) + b$ for some b with $\tau(b) = x$, so that $\tau(\sigma_x(y)) = \tau(\sigma(y - x) + b) = y - x + x = y$. Therefore, yet another description of this class is $S = y^{\#}(\mathrm{Mor}(\mathcal{S}(T^L_G)))$ (see Section 2.3), the Yoneda pullback of the class of all morphisms of sheaves on Lichtenbaum's topology T^L_G defined in Section 2.2.6, which shows that $(\mathcal{M}_G, S)$ is a quasi-abelian S-category.

1.2.4. Definition of $\mathrm{Ext}^n(A,B)$**.** Let $(\mathcal{C},S)$ be a quasi-abelian S-category. Yoneda defines $\mathrm{Ext}^n_{\mathcal{C},S}(A,B)$ for A,B in $\mathcal{C}$ to be the set[8] $\mathrm{EXT}^n(A,B)$ of exact sequences (called extensions)

$$X : 0 \to B \xrightarrow{\alpha} E_n \to \cdots \to E_1 \xrightarrow{\omega} A \to 0$$

where all the maps are in S, modulo the equivalence relation generated by maps $X \to X'$ of such extensions: commutative diagrams

$$\begin{array}{ccccccccccccc} X: & 0 & \to & B & \xrightarrow{\alpha} & E_n & \to & \cdots & \to & E_1 & \xrightarrow{\omega} & A & \to & 0 \\ & & & \| & & \downarrow & & & & \downarrow & & \| & & \\ X': & 0 & \to & B & \xrightarrow{\alpha'} & E'_n & \to & \cdots & \to & E'_1 & \xrightarrow{\omega'} & A & \to & 0 \end{array}$$

(the maps $E_i \to E'_i$ do not need to be in S). Thus two extensions X and X' are **equivalent** if and only if there is a string of such maps of extensions $X \to X_1 \leftarrow X_2 \to \cdots \leftarrow X_N \to X'$. Let us denote by $[X]$ the class of the extension X under this equivalence relation.

To describe the **addition law** for two classes of extensions [**39**, p. 537, (3.3.1)], we first need some definitions. The **cotranslation** of an extension X as above by $\alpha : A' \to A$ is the extension

$$X \bigcirc \alpha : 0 \to B \to E_n \to \cdots \to E_2 \to E'_1 \to A' \to 0$$

where $E'_1 = E_1 \times_A A'$ and the map $E_2 \to E'_1$ is the unique map which makes the composition with $E'_1 \to A'$ equal to zero and the composition with $E'_1 \to E_1$ equal to the map $E_2 \to E_1$. It turns out that, indeed, the sequence $X \bigcirc \alpha$ is exact, and the class of $X \bigcirc \alpha$ only depends on the class of X. Note that all maps in $X \circ \alpha$ are in S by (QA) and an argument by Yoneda. Therefore, we can define $[X] \circ \alpha$ to be $[X \bigcirc \alpha]$. Dually, we define the **translation** of X by $\beta : B \to B'$ to be the extension

$$\beta \bigcirc X : 0 \to B' \to E'_n \to E_{n-1} \to \cdots \to E_1 \to A \to 0$$

[8] Yoneda avoids all "metamathematical arguments" and basically thinks of categories as small so that he only needs to work with sets.

where $E'_n = B' \sqcup_B E_n$, and we define $\beta \circ [X]$ to be $[\beta \bigcirc X]$. Now the addition law states that, for X and X' as above, we have $[X] + [X'] = \nabla_B \circ [X \oplus X'] \circ \Delta_A$, where $\Delta_A : A \to A \oplus A$ is the diagonal map and $\nabla_B : B \oplus B \to B$ is the codiagonal map. The **zero element** is given by $0 \to B \to A \oplus B \to A \to 0$ for $n = 1$ and by $0 \to B \xrightarrow{\text{id}} B \xrightarrow{0} 0 \to \cdots \to 0 \to A \xrightarrow{\text{id}} A \to 0$ for $n > 1$. The **negative** of an element $[X]$ is given by switching the sign of an odd number of maps among $B \to E_n, E_n \to E_{n-1}, \ldots, E_1 \to A$.

1.2.5. A Theorem on Embeddings.

Lemma 1.2.10 (Short Five Lemma). *If we have a commutative diagram*

$$\begin{array}{ccccccccc} 0 & \to & A & \to & B & \to & C & \to & 0 \\ & & \| & & \downarrow^{\phi} & & \| & & \\ 0 & \to & A & \to & B' & \to & C & \to & 0 \end{array}$$

of proper morphisms in a quasi-abelian category $\mathcal{C}$, then ϕ is an isomorphism.

Proof. This is [**5**, Corollary 3.2]. □

Theorem 1.2.11. *Suppose $(\mathcal{C}, S)$ and $(\mathcal{D}, S')$ are quasi-abelian S-categories and $\alpha : \mathcal{C} \to \mathcal{D}$ is a fully faithful exact additive functor, i.e. α carries an exact sequence in S into an exact sequence in S'. Then for all $n \geq 0$ there are natural maps $\phi_n : \mathrm{Ext}^n_{\mathcal{C},S}(A, B) \to \mathrm{Ext}^n_{\mathcal{D},S'}(\alpha(A), \alpha(B))$ taking the class of an extension*

$$X : 0 \to B \to E_n \to \cdots \to E_1 \to A \to 0$$

to the class of the extension

$$\alpha X : 0 \to \alpha(B) \to \alpha(E_n) \to \cdots \to \alpha(E_1) \to \alpha(A) \to 0.$$

ϕ_n is an isomorphism for $n = 0$ and an injection for $n = 1$.

Now suppose that in addition ϕ_n is a surjection for $n = 1$ and either the following or its dual holds:

- *(E) For every monomorphism $m : \alpha(B) \to E$ in S' there is a monomorphism $m' : B \to E'$ in S and a map $f : E \to \alpha(E')$ in $\mathcal{D}$ such that $f \circ m = \alpha(m')$.*

Then ϕ_n is an isomorphism for all n.

Proof. First we need to show that the map ϕ_n is well-defined. Suppose we have an extension X' equivalent to X in $(\mathcal{C}, S)$. Then there is a string of commutative diagrams connecting X to X', and if we apply α to the commutative diagrams, we get commutative diagrams in $\mathcal{D}$ (because α is a functor), so $\alpha(X)$ is equivalent to $\alpha(X')$. Since α is exact and preserves direct sums (because α is additive), it is easy to see that α commutes with translation and cotranslation, so ϕ_n is a homomorphism of abelian groups.

Of course, the statement that ϕ_0 is an isomorphism of abelian groups is equivalent to the statement that α is fully faithful and additive. To show ϕ_1 is injective, suppose we have two extensions

$$X : 0 \to B \to E \to A \to 0$$

$$X' : 0 \to B \to E' \to A \to 0$$

in $(\mathcal{C}, S)$ and $\alpha(X)$ is equivalent to $\alpha(X')$. Then there is a string of commutative diagrams in $\mathcal{D}$ connecting them. But by Lemma 1.2.10, any such commutative diagram induces an isomorphism on the middle terms. Thus, by composing the isomorphisms on the middle terms and using the commutativity in the commutative diagrams connecting $\alpha(X)$ and $\alpha(X')$, we get an isomorphism $\alpha(E) \xrightarrow{\sim} \alpha(E')$ making the diagram $\alpha(X) \to \alpha(X')$ commutative. The fact that α is fully faithful then gives us a map $X \to X'$, so the two are equivalent.

Now we assume that ϕ_1 is surjective and (E) holds (if the dual of (E) holds, we can use the dual of the following proof). We first show surjectivity of ϕ_n for $n > 1$; this argument is due to Theo Bühler[9]. Let

$$X : 0 \to \alpha(B) \xrightarrow{e_0} E_1 \xrightarrow{e_1} E_2 \to \cdots \to E_n \xrightarrow{e_n} \alpha(A) \to 0$$

be an extension in $(\mathcal{D}, S')$. Let I_k be the image of e_k, so e_k decomposes as $E_k \xrightarrow{q_k} I_k \xrightarrow{i_k} E_{k+1}$ for $k = 1, \cdots, n-1$. We construct the diagram below as follows. By (E) we

[9]Private communication.

obtain the maps $e_0' : B \to E_1'$ and $s_1 : E_1 \to \alpha(E_1')$, then take the cokernel $q_1' : E_1' \to I_1'$ and note that there is an induced map $t_1 : I_1 \to \alpha(I_1')$ such that $t_1 \circ q_1 = \alpha(q_1') \circ s_1$, since q_1 is the cokernel of e_0 and $\alpha(q_1') \circ s_1 \circ e_0 = 0$. Next, we take the fibered coproduct $F_2 = \alpha(I_1') \sqcup_{I_1} E_2$ in $\mathcal{D}$. Note that, by axiom (Q2*), the map f_1 is a monomorphism in S', g_2 is an epimorphism in S', and we have commutativity so far, i.e. q_2 is the composition $E_2 \to F_2 \xrightarrow{g_2} I_1$. Now we use (E) again to obtain the maps i_1' and q_2', take the fibered coproduct $F_3 = \alpha(I_2') \sqcup_{I_2} E_3$, and keep going in a similar fashion.

$$\begin{array}{ccccccccccccc}
\alpha(B) & \xrightarrow{e_0} & E_1 & \xrightarrow{q_1} & I_1 & \xrightarrow{i_1} & E_2 & \xrightarrow{q_2} & I_2 & \xrightarrow{i_2} & E_3 & \cdots \\
\| & & \downarrow & & \downarrow & & \downarrow & & \| & & \| & \\
\alpha(B) & \xrightarrow{\alpha(e_0')} & \alpha(E_1') & \xrightarrow{\alpha(q_1')} & \alpha(I_1') & \xrightarrow{f_1} & F_2 & \xrightarrow{g_2} & I_2 & \xrightarrow{i_2} & E_3 & \cdots \\
 & & & & \| & & \downarrow & & \downarrow & & \downarrow & \\
 & & & & \alpha(I_1') & \xrightarrow{\alpha(i_1')} & \alpha(E_1') & \xrightarrow{\alpha(q_2')} & \alpha(I_2') & \xrightarrow{f_2} & F_3 & \cdots
\end{array}$$

At the end, when we take the fibered coproduct $F_n = \alpha(I_{n-1}') \sqcup_{I_{n-1}} E_n$, we get an exact sequence $0 \to \alpha(I_{n-1}') \to F_n \to \alpha(A) \to 0$ in S', and by the surjectivity of ϕ_1 and the Short Five Lemma we see that $F_n \cong \alpha(E_n')$ for some $E_n' \in \mathcal{C}$. Thus there is a commutative diagram linking X to $\alpha(X')$, with

$$X' : 0 \to B \xrightarrow{e_0'} E_1' \xrightarrow{i_1' \circ q_1'} E_2' \to \cdots \to E_n' \to A \to 0$$

and surjectivity of ϕ_n is proven.

Yoneda's Similarity Theorem II [**39**, p. 575] states that if we have a sequence

$$X : 0 \to B \to E_1 \to \cdots \to E_n \to A \to 0$$

then $[X] = 0$ if and only if there is an exact sequence X' and a commutative diagram

$$\begin{array}{cccccccccccccccc}
X: & 0 & \to & B & \to & E_1 & \to & \cdots & \to & E_{n-1} & \to & E_n & \to & A & \to & 0 \\
 & & & \| & & \downarrow & & & & \downarrow & & \| & & & & \\
X': & 0 & \to & B & \to & E_1' & \to & \cdots & \to & E_{n-1}' & \to & E_n & \to & 0 & &
\end{array}$$

To prove injectivity, we use the proof of surjectivity and both implications of this Similarity Theorem II: if $[\alpha(X)] = 0$ then we get a commutative diagram

$$\begin{array}{ccccccccccccc}
\alpha(X): & \alpha(B) & \to & \alpha(E_1) & \to & \cdots & \to & \alpha(E_{n-1}) & \to & \alpha(E_n) & \to & \alpha(A) \\
& \| & & \downarrow & & & & \downarrow & & \| & & \\
X': & \alpha(B) & \to & E'_1 & \to & \cdots & \to & E'_{n-1} & \to & \alpha(E_n) & \to & 0 \\
& \| & & \downarrow & & & & \downarrow & & \| & & \\
\alpha(X''): & \alpha(B) & \to & \alpha(E''_1) & \to & \cdots & \to & \alpha(E''_{n-1}) & \to & \alpha(E_n) & \to & 0
\end{array}$$

and the map $\alpha(X) \to \alpha(X'')$ comes from a map $X \to X''$ since α is fully faithful, hence $[X] = 0$. □

1.2.6. Universality of $\mathrm{Ext}^n_{C,S}(A, B)$**.** Let (C, S) be a quasi-abelian S-category. Suppose $(h^i)_{i=0}^{\infty}$ is a sequence of covariant functors from C to **Ab** that form an **exact connected sequence of functors** with respect to the class S, i.e. such that for every short exact sequence $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ with $\alpha, \beta \in S$ we get a long exact sequence

$$\cdots \to h^i(A) \to h^i(B) \to h^i(C) \to h^{i+1}(A) \to \cdots$$

which is functorial in the short exact sequence (the morphisms connecting the short exact sequences do not need to be in S). The notion of an exact connected sequence of functors with respect to S only differs from the notion of a cohomological functor [**10**] in that only short exact sequences in S need to give a long exact sequence. An exact connected sequence (h^i) of functors as above is **universal** if for any other such exact connected sequence of functors $(\tilde{h}^i)$ and any natural transformation $\eta_0 : h^0 \to \tilde{h}^0$ there exist unique natural transformations $\eta_i : h^i \to \tilde{h}^i$ such that for any short exact sequence in S as above, the diagram

$$\begin{array}{ccc}
h^i(C) & \to & h^{i+1}(A) \\
\downarrow{\scriptstyle \eta_i} & & \downarrow{\scriptstyle \eta_{i+1}} \\
\tilde{h}^i(C) & \to & \tilde{h}^{i+1}(A)
\end{array} \tag{2}$$

commutes.

Theorem 1.2.12 (D. Wigner). *Let* $(\mathcal{C}, S)$ *be a quasi-abelian S-category and Z an object in* $\mathcal{C}$. *Then* $(\mathrm{Ext}^i_{\mathcal{C},S}(Z,-))_{i=0}^{\infty}$ *forms a universal exact connected sequence of functors.*

Proof. The proof is given in [**36**, pp. 11-14] for $S =$ the class of proper morphisms in $\mathcal{M}_G$, but it works in this generality. □

Theorem 1.2.13 (D. Wigner). *Let* $(\mathcal{C}, S)$ *be a quasi-abelian S-category, and suppose* (H^i) *and* (h^i) *are two exact connected sequences of functors, with* (H^i) *universal, such that* $H^0 \cong h^0$ *and*

(C) for $i > 0$ *and* $x \in h^i(A)$ *there is a proper monomorphism* $\theta : A \to B$ *such that* $\theta_*(x) = 0 \in h^i(B)$.

Then the maps $H^i \to h^i$ *given by universality are isomorphisms.*

Proof. The proof is given in [**36**, pp. 14-15]. □

1.3. Topological Groups and Associated Categories

1.3.1. Topological Groups. Throughout this section, G will be a topological group. Unless explicitly mentioned otherwise, topological groups, and more generally topological spaces, do not have to satisfy T_1, i.e. the one-point sets do not have to be closed. We recall a couple of facts about topological groups from [**26**].

First, for a group G with a topology to be a topological group, that is, for the maps $G \times G \to G : (g,h) \mapsto gh$ and $G \to G : g \mapsto g^{-1}$ to be continuous, it is necessary and sufficient that for any $a, b \in G$ and any neighborhood U of ab^{-1} there exist a neighborhood U_a of a and a neighborhood U_b of b such that $U_a U_b^{-1} \subseteq U$ ($U^{-1} = \{x^{-1} \mid x \in U\}$). In fact, by induction, given any neighborhood U of an element of the form $x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}$, we can find neighborhoods U_1 of $x_1, \ldots, U_k$ of x_k such that $U_1^{n_1} U_2^{n_2} \cdots U_k^{n_k} \subseteq U$ (the powers n_i are allowed to be negative). We will use this fact often throughout.

Second, any topological group is a **regular** space, i.e. for any $g \in G$ and any neighborhood U of g, there exists a neighborhood V of g such that $\overline{V} \subseteq U$. Left and right multiplication by any $g \in G$ are homeomorphisms of G, so it is enough to show this for $g = e$. By the previous paragraph, there exists a neighborhood V of e such that $VV^{-1} \subseteq U$. Now if $a \in \bar{V}$ then any neighborhood of a, in particular aV, intersects V, i.e. there exist $b, c \in V$ such that $ab = c$, so $a = cb^{-1} \in VV^{-1} \subseteq U$. It is easy to show this condition for regularity is equivalent to the usual one, that any point x and a closed set F not containing x can be separated by disjoint open sets. Thus topological groups provide a source of examples of regular spaces that are not necessarily T_1. Of course, a T_0 regular space is T_3, which implies T_2 (Hausdorff), which implies T_1. Recall T_0, or **Kolmogorov**, means any two points are topologically distinguishable, that is, for any $x \neq y$, there is an open set U containing one of x, y, but not both.

Note that for any subgroup H of G, the quotient map $q : G \to G/H$ is open if we give G/H the quotient topology, because for any open set $U \subseteq G$, $q^{-1}(q(U)) = UH = \bigcup_{h \in H} Uh$ is the union of open sets, hence $q^{-1}(q(U))$ is open, hence $q(U)$ is open. We will often use this fact.

1.3.2. The Category C_G of G-Spaces. Let G be a topological group. We will be working a great deal with the category C_G of G-spaces, so we collect a few facts about them here. The objects are G-sets X with a topology such that the map $G \times X \to X : (g, x) \mapsto g \cdot x$ is continuous. The morphisms are continuous G-maps. The initial object is $\emptyset$, and the final object is pt.

1. Arbitrary products exist. Indeed, let $(X_i)_{i \in I}$ be a collection of G-spaces. Then $\prod X_i$, with the product topology and diagonal G-action, is the categorical product in C_G. To verify that the action of G is continuous, i.e. that the map $p : G \times \prod X_i \to \prod X_i$ is continuous, first let U be a subset of $\prod X_i$ which is in the basis for the topology of $\prod X_i$. In other words, without loss of generality we can take U_i to be an open subset of X_i for $i = i_1, \ldots, i_n$ and set $U = U_1 \times \cdots \times U_n \times \prod_{i \notin \{i_1, \ldots i_n\}} X_i$. We want to show $p^{-1}(U)$ is an open subset of $G \times \prod X_i$. Let $m_i : G \times X_i \to X_i : (g, x) \mapsto g \cdot x$. Then

$$p^{-1}(U) = \{(g \in G, (x_i \in X_i)) \mid g \cdot x_i \in U_i \forall i = i_1, \ldots, i_n\}$$

$$= \bigcap_{i=1}^{n} \{(g \in G, (x_i \in X_i)) \mid g \cdot x_i \in U_i\}$$
$$= \bigcap_{i=1}^{n} m_i^{-1}(U_i) \times \prod_{j \neq i} X_j$$

is a finite intersection of open sets, hence is open. Since we know $\prod X_i$ satisfies the universal property in both the category of G-sets and the category of topological spaces, it satisfies the universal property in C_G. The same argument works for all such verifications, so we will skip it.

2. Arbitrary coproducts exist: take $(X_i)_{i \in I}$ as above. Then $\coprod X_i$, with the disjoint union topology (a set $U \subseteq \coprod X_i$ is open if and only if $U \cap X_i$ is open for all X_i), is the categorical coproduct in C_G. To verify that the map $p : G \times \coprod X_i \to \coprod X_i$ is continuous, take an open set $U \subseteq \coprod X_i$ and let m_i be as above. Then $p^{-1}(U) \cap (G \times X_i) = m_i^{-1}(U)$ is open, hence $p^{-1}(U)$ is open.

3. A G-subset (i.e. a subset that is closed under the action of G) X of a G-space Y, with the subspace topology, has continuous G-action: let $p : G \times Y \to Y$ and $q : G \times X \to X$ be the two actions, and let U be an open subset of X. Then $q^{-1}(U) = \{(g, x) \mid g \cdot x \in U\} = p^{-1}(U)$ because if $y \in Y$ and $g \cdot y = u \in U$ then $y = g^{-1} \cdot u \in X$ since X is a G-subset, so $q^{-1}(U)$ is open in $G \times X$.

4. Equalizers exist: given maps $f, g : X \to Y$, we have $X \times_Y X = \{(x, x') \mid f(x) = g(x')\}$, with the subspace topology of $X \times X$ and diagonal action; since products of G-spaces are G-spaces and G-subsets of G-spaces are G-spaces, $X \times_Y X$ is a G-space. This implies arbitrary small inverse limits, in particular fibered products, exist in C_G.

Proposition 1.3.1. *Suppose $\sim$ is an equivalence relation on the G-space Y such that $x \sim y \Rightarrow g \cdot x \sim g \cdot y$ for all $g \in G$. Let $X = Y/\sim$ with the quotient topology and G-action given by $g \cdot [x] = [g \cdot x]$, where $[x]$ is the equivalence class represented by x. If the quotient map $Y \to X$ is open, then X is a G-space.*

Proof. Since $\mathrm{id} : G \to G$ and $q : Y \to X$ are surjective open quotient maps, their product $\mathrm{id} \times q$ is a surjective open quotient map. Now (see the diagram below) given an open subset U of X, the preimage $V = p^{-1}(U)$ under the map $G \times X \xrightarrow{p} X$ is open because the preimage W of V under the map $\mathrm{id} \times q$ is precisely $m^{-1}(q^{-1}(U))$ (where $m : G \times Y \to Y$ is the action of G on Y), which is open.

$$\begin{array}{ccc} W \subset G \times Y & \xrightarrow{m} & Y \\ {\scriptstyle \mathrm{id} \times q} \downarrow & & {\scriptstyle q} \downarrow \\ V \subset G \times X & \xrightarrow{p} & X \supset U \end{array}$$ □

For example, this implies immediately that for any subgroup H of G, G/H (with the quotient topology from G) is a continuous G-space, since the quotient map $q : G \to G/H$ is open by the argument in Section 1.3.1.

5. Coequalizers do not always exist, but given open maps $f, g : X \rightrightarrows Y$ in C_G, the coequalizer $Y \xrightarrow{q} Z$ is given by $Z = Y/\sim$, where $\sim$ is the equivalence relation generated by identifying $f(x) \sim g(x)$ for all $x \in X$, and q is the quotient map. The set $Y/\sim$ has the G-action $g \cdot [y] = [g \cdot y]$. This is well-defined: if $y = f(x)$ and $y' = g(x)$ then for any $g_0 \in G$ we have

$$g_0 \cdot y = g_0 \cdot f(x) = f(g_0 \cdot x) \sim g(g_0 \cdot x) = g_0 \cdot g(x) = g_0 \cdot y'.$$

In general, if $y \sim y'$ then there is a finite string of equivalences connecting them, and by induction $g_0 \cdot y \sim g_0 \cdot y'$. As a space, Z has the quotient topology. It is well-known that this space has the universal property as a topological space, and it is easy to check that it satisfies the universal property in the category of G-sets. To see that the quotient map $q : Y \to Z$ is open, note that, for any $y \in Y$,

$$\{y' \in Y \mid y' \sim y\} = f(g^{-1}(y)) \cup g(f^{-1}(y)) \cup f(g^{-1}(f(g^{-1}(y)))) \cup f(g^{-1}(g(f^{-1}(y)))) \cup \cdots$$

so for an open set $U \subseteq Y$, $q^{-1}(q(U)) = \{y \in Y \mid \exists u \in U, y \sim u\}$ is the union of $f(g^{-1}(U)), g(f^{-1}(U))$, etc., each of which is open. Proposition 1.3.1 shows that Z has continuous G-action.

7. There is an inclusion functor from topological spaces to G-spaces taking a topological space X to the G-space X with trivial G-action (the map $G \times X \to X$ is

continuous since the preimage of an open set U is $G \times U$). Clearly, the G-set $X \times G$ with G-action $g(x, g') = (x, gg')$ has continuous G-action; such G-spaces will be used in Chapter 2.

To become more familiar with G-spaces, let us first explore what topologies are possible for G-spaces which consist of a single orbit.

Proposition 1.3.2. *Let X be a G-space and $\mathcal{O} \cong G/H$ the orbit of $x \in X$ in X ($H = \mathrm{Stab}(x)$). Then the subspace topology on $\mathcal{O}$ must be no finer than the quotient topology.*

Proof. The map $f : G \to X : g \mapsto g \cdot x$ is continuous, so if U is an open set of $\mathcal{O}$ which is the intersection of an open set V of X with $\mathcal{O}$, then $f^{-1}(V) = f^{-1}(U)$ is open, which means U is open in the quotient topology. □

Proposition 1.3.3. *Let G be a topological group, let H be a subgroup of G, and let $\mathcal{O} = G/H$ as a G-set. Let G' be a topological group whose underlying group is G and whose topology is coarser than that of G. Then the topology induced on $\mathcal{O}$ by the quotient map $q : G' \to \mathcal{O}$ makes $\mathcal{O}$ into a G-space.*

Proof. We must show that the map $G \times \mathcal{O} \xrightarrow{a} \mathcal{O}$ is continuous. Let $U \subseteq \mathcal{O}$ be an open subset, and let $\bar{U} := q^{-1}(U) \subseteq G'$. Let $(g_1, g_2H) \in a^{-1}(U)$. Then the preimage of $\bar{U}$ under $G' \times G' \to G'$ contains (g_1, g_2) and is open, so there are open subsets $U_1, U_2 \subseteq G'$ such that $g_1 \in U_1, g_2 \in U_2, U_1U_2 \subseteq \bar{U}$. Since q is an open map, $q(U_2)$ is an open subset of $\mathcal{O}$, and since the topology of G' is coarser than that of G, U_1 is open in G, hence $U_1 \times q(U_2)$ is a neighborhood of (g_1, g_2H) in $a^{-1}(U)$. This means $a^{-1}(U)$ can be covered by open subsets, hence is open. □

Next, let us explore what is necessary for a discrete G-set to have a continuous G-action.

Proposition 1.3.4. *For any topological group G and G-set X with discrete topology, the map $G \times X \xrightarrow{f} X : (g, x) \mapsto g \cdot x$ is continuous if and only if the stabilizer $\mathrm{Stab}(x) = \{g \in G \mid g \cdot x = x\}$ of $x \in X$ is open for all $x \in X$.*

Proof. In one direction, for any $x \in X$, the map $G \to X : g \mapsto g \cdot x$ is continuous, hence the preimage $\mathrm{Stab}(x)$ of x is open. Conversely, suppose all the stabilizers are open. To show f is continuous, it is enough to show that the preimage $f^{-1}(x)$ of a single point $x \in X$ is open. Now $f^{-1}(x) = \bigcup_{y \in X}\{g \in G \mid gy = x\}$, so it is enough to show each set $H = \{g \in G \mid gy = x\}$ is open. If it is empty, we are done. Otherwise, suppose $h \in H$. Then $H = \mathrm{Stab}(x)h$: given $g \in H$, we have $g = (gh^{-1})h$ with $gh^{-1} \in \mathrm{Stab}(x)$, and conversely given $g \in \mathrm{Stab}(x)$, $ghy = gx = x$. Since $\mathrm{Stab}(x)$ is open, so is $H = \mathrm{Stab}(x)h$ (multiplication by h is a homeomorphism on G). □

Finally, if G is given the two special topologies possible for any topological group, the discrete one and the trivial one, we explore what topologies make a G-set into a G-space.

Proposition 1.3.5. *Let G be a discrete group and X a G-set which is also a topological space. The action of G is continuous if and only if $g \cdot U$ is open for all open sets $U \subseteq X$ and $g \in G$.*

Proof. If the action of G is continuous then multiplication by g is a homeomorphism, so $g \cdot U$ is open for all open sets $U \subseteq X$ and $g \in G$. For the converse, the preimage of an open set $U \subseteq X$ under the action $G \times X \to X$ is $\{(g, x) \mid g \cdot x \in U\} = \bigcup_{g \in G}\{g\} \times g^{-1}U$ is the union of open sets, hence open. □

Proposition 1.3.6. *Let G be a group with trivial topology and X a G-set which is also a topological space. The action of G is continuous if and only if each open set of X is a union of orbits.*

Proof. If the action $G \times X \xrightarrow{a} X$ is continuous, then by Lemma 1.3.2, the subspace topology for any orbit must be the trivial topology, which means for any open set U of X and any orbit $\mathcal{O}$, either $U \cap \mathcal{O} = \emptyset$ or $U \cap \mathcal{O} = \mathcal{O}$. Conversely, if each open set U of X is a union of orbits, then $a^{-1}(U) = G \times U$ is open. □

Proposition 1.3.7. *There is a left-adjoint Q and a right-adjoint S to the forgetful functor R from $\mathcal{C}_G$ to G-**set** .*

Proof. For a G-set X, define a space $Q(X) = X$ with the finest topology T making X a G-space (see Lemma 1.3.12). Explicitly, give each orbit $\mathcal{O} \cong G/H$ of X the quotient topology and define a set $U \subseteq X$ to be open if and only if its intersection with each orbit $\mathcal{O}$ is open. Clearly, a G-space topology on X cannot be finer because the orbits cannot have any more open sets as subspaces by Proposition 1.3.2, and this makes X the disjoint union of the orbits, each of which has the finest topology possible. For a G-map $f : X \to Y$ of G-sets, define $Q(f) = f : Q(X) \to Q(Y)$. Then $Q(f)$ is continuous because $Q(Y)$ is a G-space, so the weak topology on X with respect to f (see Lemma 1.3.13) is a G-space topology, hence coarser than the topology of $Q(X)$. For the same reason, there is a natural continuous map $QRX \xrightarrow{\text{id}} X$ for any G-space X. Now $RQ = \text{id}$, so $Q \to QRQ \to Q$ and $R \to RQR \to R$ are isomorphisms, hence the functors are adjoint (of course, one can also directly show that $\text{Hom}_{C_G}(QX, Y) = \text{Hom}_G(X, RY)$).

For the right-adjoint, let $S(X) = X$ with the trivial topology and $S(X \xrightarrow{f} Y) = f$ (any map into a space with trivial topology is continuous). Again, there is a natural continuous map $X \xrightarrow{\text{id}} SRX$ for any G-space X and $RS = \text{id}$, so the two are adjoint. □

1.3.3. Topological G-Modules.

1.3.3.1. *$\mathcal{M}_G$ is Quasi-Abelian.* A **topological G-module** A is an abelian topological group which is also a G-module, such that the action $G \times A \to A$ is continuous. In this section, we will be working with the category $\mathcal{M}_G$ of (not necessarily Hausdorff) topological G-modules, where the morphisms are continuous G-equivariant homomorphisms of G-modules.

First we note that $\mathcal{M}_G$ is an additive, in fact G-linear, category (that is, each $\text{Hom}(A, B)$ set is a G-module). The G-module $\{0\}$ is the zero object, and for any two topological G-modules A and B, the G-module $A \oplus B$ with the product topology is clearly the product, and also the direct sum. Indeed, the inclusions $A \hookrightarrow A \oplus B$ and $B \hookrightarrow A \oplus B$ are continuous (being the product of the identity and the zero map), and given two maps $f : A \to C, f' : B \to C$ in $\mathcal{M}_G$, the map $A \oplus B \to C : (a, b) \mapsto f(a) + f(b)$

is continuous, being the composition $A \oplus B \xrightarrow{(f,f')} C \oplus C \xrightarrow{+} C$. Similarly, if we have two morphisms $f, f' : A \to B$ then the morphism $f + f' : a \mapsto f(a) + f'(a)$ is continuous, the map $-f : a \mapsto -f(a) = f(-a)$ is continuous, and for any $g \in G$ the map $(g \cdot f)(a) = g \cdot f(a)$ is continuous. This makes $\mathrm{Hom}(A, B)$ into a G-module (of course the zero map is still continuous), and composition of morphisms is bilinear.

Note that these arguments also work for the category $\mathcal{M}_G^{ne}$ of topological G-modules where the maps are continuous homomorphisms which are *not* (G)-*equivariant* (hence the notation "ne"), so $\mathcal{M}_G^{ne}$ is a G-linear category.

The **kernel** of a map $f : A \to B$ in $\mathcal{M}_G$ is the usual kernel $k : K \hookrightarrow A$, where $K = \{a \in A \mid f(a) = 0\}$, with the subspace topology. It is easy to check that K is a topological G-module, and that it satisfies the universal property. The **cokernel** of a map $A \xrightarrow{f} B$ is the quotient map $c : B \to C$, where $C = B/f(A)$ with the quotient topology. The map c is open, so C is a G-space, and it is easy to see that C is a topological G-module which satisfies the universal property. This implies that fibered products and fibered coproducts in $\mathcal{M}_G$ exist and are exactly what we would expect. The same reasoning applied for $G = 1$ shows that, as long as a map $f : A \to B$ is G-equivariant, it has a kernel and a cokernel in $\mathcal{M}_G^{ne}$.

In $\mathcal{M}_G$, a map $f : A \to B$ is **proper** (see Section 1.2.1) if and only if it is open as a map onto its image: $\mathrm{coker}(\ker f) = A/\ker f$ and $\ker(\mathrm{coker}\, f) = \mathrm{Im}(f)$; as a map of G-modules, $\phi : A/\ker f \to \mathrm{Im}(f)$ is an isomorphism, and the map is continuous because f is continuous, so in order for ϕ to be a homeomorphism we need ϕ to be open, which is the same as saying f is open as a map onto its image. An injective map in $\mathcal{M}_G$ is proper if and only if it is a homeomorphism onto its image; thus, proper monomorphisms, which are kernels of their own cokernels, are precisely the maps which are homeomorphisms onto their image. A surjective map in $\mathcal{M}_G$ is proper if and only if it is open; thus, proper epimorphisms, which are cokernels of their own kernels, are precisely the open continuous surjective maps. The same is true for a G-map in $\mathcal{M}_G^{ne}$.

A sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ of topological G-modules is **exact** if and only if the sequence is exact a sequence of G-modules, f is a homeomorphism onto its image, and g is open (if g is a quotient map, then it is automatically open). Note that if C is Hausdorff then $A = g^{-1}(\{0\})$ is a closed subgroup of C.

Proposition 1.3.8. *$\mathcal{M}_G$ is a quasi-abelian category.*

Proof. The pullback of a proper epimorphism is a proper epimorphism, since openness and surjectivity are preserved under pullback even in the category of topological spaces (see the proof of Proposition 2.2.8). We just have to show that the pushout of a proper monomorphism is a proper monomorphism. To see this, suppose $f : A \hookrightarrow B$ is a proper monomorphism and $g : A \to C$ is any map in $\mathcal{M}_G$. Then $B \sqcup_A C = (B \oplus C)/H$, where $H = \langle f(a), -g(a) \mid a \in A \rangle$. The pushout $f' : C \to B \sqcup_A C$ is injective because if $f'(c) \in H$ then $c = -g(a)$ for some $a \in A$ such that $f(a) = 0$, but f is injective so $a = 0$, hence $c = 0$.

We now show that f' is open as a map onto its image. Let $q : B \oplus C \to B \sqcup_A C$ be the quotient map (which is an open map), U be an open set in C, and $u_0 \in U$. There is a neighborhood V of 0 in C such that $V + V \subseteq U - u_0$. Since f is a homeomorphism onto its image, there is an open set W in B with $W \cap f(A) = f(g^{-1}(V))$. I claim that $q(W \times (u_0 + V)) \cap f'(C) \subseteq f'(U)$, so $q(W \times (u_0 + V))$ is a neighborhood of $f'(u_0)$ whose intersection with $f'(C)$ is contained in $f'(U)$. This implies $f'(U)$ is open in $f'(C)$. To verify this, suppose $w \in W, v \in V$ and $q(w, u_0 + v) \in f'(C)$. Then there exists $c \in C$ and $a \in A$ such that $(w, u_0 + v) = (f(a), c - g(a))$. But if $w = f(a)$ then $w \in f(A)$ and $W \cap f(A) = f(g^{-1}(V))$, so $f(a) \in f(g^{-1}(V))$, which implies $g(a) = v' \in V$ since f is injective. Then

$$u_0 + v = c - g(a) \Rightarrow c = u_0 + v + g(a) = u_0 + v + v'.$$

But $u_0 + V + V \subseteq U$, so $c \in U$, i.e. $q(w, u_0 + v) \in f'(U)$, as desired. □

1.3.3.2. *The Categories of Topological G-modules.*

(1) $\mathcal{M}_G$ is the category of all topological G-modules,

(2) $\mathcal{M}_G^{pm}$ is the category of all pseudometrizable topological G-modules (i.e. those whose topology is induced by some pseudometric - see below for the definition),

(3) $\mathcal{M}_G^{H}$ is the category of Hausdorff G-modules,

(4) $\mathcal{M}_G^{m}$ is the category of metrizable G-modules,

(5) $\mathcal{M}_G^{cm}$ is the category of completely metrizable G-modules (i.e. those whose topology is induced by some complete metric), and

(6) $\mathcal{M}_G^{P}$ is the category of **Polish** G-modules, i.e. second countable[10] completely metrizable G-modules.

A **pseudometric**[11] on a set X is a function $d : X \times X \to \mathbb{R}_{\geq 0}$ such that

(1) $d(x, x) = 0$ for all $x \in X$,

(2) $d(x, y) = d(y, x)$ for all $x, y \in X$,

(3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

A pseudometric induces a topology on X just as a metric does: a basis for the topology is the collection of open balls $B(x, \varepsilon) = \{y \in X \mid d(x, y) < \varepsilon\}$ for $x \in X, \varepsilon \in \mathbb{R}_{>0}$. The only difference between a pseudometric and a metric is that for a pseudometric d, we could have $d(x, y) = 0$ with $x \neq y$.

We showed in Section 1.3.3 that $\mathcal{M}_G$ is quasi-abelian. In fact, all the categories mentioned above are quasi-abelian. First note that *finite products exist in all the categories*: the product of two Hausdorff spaces is Hausdorff; the product of two pseudometric spaces is pseudometrizable, with pseudometric given by the sum of the two pseudometrics; thus the product of two metric spaces is metrizable; the product of two complete metric spaces is complete; and the product of two second countable spaces is second countable. Second, note that *kernels exist*: the kernel in $\mathcal{M}_G$ for any map in the above categories is also in the respective category: a subspace of a Hausdorff (resp. pseudometrizable, metrizable, second countable)

[10] A topological space is second countable if there is a countable base for the whole space.

[11] It is also sometimes called a **semimetric** [**37**], since it is so closely related to the notion of seminorm.

space is Hausdorff (resp. pseudometrizable, metrizable, secound countable), and in $\mathcal{M}_G^{cm}$ the kernel is a closed subspace, which is also complete. Thus the fibered products in these categories exist and are the same as in $\mathcal{M}_G$.

However, *cokernels do not always exist*. For instance, in $\mathcal{M}_G^H$, $\mathcal{M}_G^m$, $\mathcal{M}_G^{cm}$, and $\mathcal{M}_G^p$ the cokernel of a map $f : A \to B$ exists if and only if the image of A is closed in B, though in $\mathcal{M}_G^{pm}$, cokernels do always exist (see Lemmas 1.3.10 and 1.3.9 below, and note that the quotient topology of a second countable topological group is second countable). To see that cokernels do not always exist in $\mathcal{M}_G^H, \mathcal{M}_G^m, \mathcal{M}_G^{cm}$, and $\mathcal{M}_G^p$, note that if $f : A \to B$ is a map that has a cokernel $c : B \to C$ then c is the cokernel of its own kernel $K \to B$, where $K = c^{-1}(0)$ viewed as a subspace of B. Because c is continuous and C is Hausdorff, $c^{-1}(0)$ is closed in B, i.e. the image of f is closed in B. This means f does *not* have a cokernel if its image is not closed in B.

In light of all of this, in $\mathcal{M}_G$ and $\mathcal{M}_G^{pm}$ a morphism $f : A \to B$ is **proper** if and only if it is open as a map onto its image, and in $\mathcal{M}_G^H, \mathcal{M}_G^m, \mathcal{M}_G^{cm}$, and $\mathcal{M}_G^p$, f is proper if and only if it is open as a map onto its image and the image of f is closed in B. This is because f must have a cokernel, so the image of f must be closed; conversely, if the image of f is closed and f is open onto its image, then $\ker(\operatorname{coker} f)$ exists; $\operatorname{coker}(\ker f)$ exists because the image of $\ker f$ is $f^{-1}(0)$, which is closed because f is continuous and B is Hausdorff; and the map $\operatorname{coker}(\ker f) \to \ker(\operatorname{coker} f)$ is an isomorphism. **Proper epimorphisms** are cokernels, hence just the open continuous surjective maps, and **proper monomorphisms** are kernels, i.e. injective maps which are homeomorphisms onto their image, which must be closed if they are in $\mathcal{M}_G^H, \mathcal{M}_G^m, \mathcal{M}_G^{cm}$, or $\mathcal{M}_G^p$, i.e. closed embeddings.

The pullback of a proper epimorphism is a proper epimorphism and always exists by the reasoning for $\mathcal{M}_G$, so these categories satisfy (Q2). The pushout of a proper monomorphism is a homeomorphism onto its image by the proof of Proposition 1.3.8 and thus always exists in $\mathcal{M}_G^{pm}$, so $\mathcal{M}_G^{pm}$ satisfies (Q2*). In the subcategories of $\mathcal{M}_G^H$, the image of a proper monomorphism $f : A \to B$ is closed in B. For any map $h : A \to C$, the pushout $\tilde{f} : C \to B \sqcup_A C$ of f has closed image because the pullback of $\tilde{f}(C)$ to $B \oplus C$ is $f(A) \times C$, which is closed. But since C is Hausdorff, $\{0\}$ is closed in C and C is a closed subgroup of $B \sqcup_A C$, so $\{0\}$ is closed in $B \sqcup_A C$, i.e. $B \sqcup_A C$ is Hausdorff. This shows that the pushout of a proper monomorphism exists and is a proper monomorphism in $\mathcal{M}_G^H, \mathcal{M}_G^m, \mathcal{M}_G^{cm}$, and $\mathcal{M}_G^P$. Hence all of these categories satisfy (Q2*) and thus are quasi-abelian.

Lemma 1.3.9. *If B is a subgroup of the topological abelian group A, and the topology of A is induced by a translation-invariant pseudometric d, then the topology of A/B is induced by a translation-invariant pseudometric d' given by $d'(x + B, y + B) = \inf_{b \in B} d(x + b, y)$.*

Proof. This is [**37**, Lemma 12.3.1]. See also Examples 1 and 2 on p. 238, p. 239 in [**37**]. □

Lemma 1.3.10. *If B is a closed subgroup of a complete metric G-module E, then the quotient $I = E/B$ is a complete metric G-module.*

Proof. I has a pseudometric induced from E by Lemma 1.3.9, and I is Hausdorff since B is a closed subgroup. We just need to show I is complete; the proof is basically the same as the corresponding proof when metric completeness is replaced by completeness with respect to the two-sided uniformity [**4**, Chapter IX, §3.1, Proposition 4]. Suppose $(\overline{x_n})$ is a Cauchy sequence in I. By passing to a subsequence if necessary, we can assume that for all $n = 1, 2, \dots$ for all $p, q \geq n$ we have $d(\overline{x_p}, \overline{x_q}) < 2^{-n}$. This means for all $p, q \geq n$ and all $y \in \overline{x_p}, z \in \overline{x_q}$ there exists $b \in B$ such that $d(y, z + b) < 2^{-n}$. Now we choose a sequence (x_n) in E as follows: let x_1 be any element in $\overline{x_1}$ and inductively define x_{n+1} for $n \geq 1$ to be an element in $\overline{x_{n+1}}$ such that $d(x_n, x_{n+1}) < 2^{-n}$. Then, by induction, $d(x_n, x_{n+p}) < 2^{-n} + 2^{-(n+1)} + \cdots + 2^{-(n+p-1)} < 2^{1-n}$. Thus (x_n) is a Cauchy sequence in E, hence converges to some $x \in E$, and since the quotient map $E \to I$ is continuous, $(\overline{x_n})$ converges to $\bar{x}$. □

1.3.3.3. *Adjoints, Injectives and Projectives.*

Theorem 1.3.11. *The forgetful functor $\mathcal{M}_G \to C_G$ has a left adjoint F. The abelian group underlying FX for $X \in C_G$ is the free abelian group on X. All limits and colimits exist in $\mathcal{M}_G$, and the underlying abelian group is the limit or colimit of the G-modules.*

Proof. This follows from the general work of Oswald Wyler [**38**]. □

For explicit descriptions of the topology of FX for $X \in C_G$ and much discussion on this topic, the standard references are [**19**] and [**9**]. An explicit description is certainly nontrivial. For example, on a related note, finding an explicit description of the topology of a colimit in $\mathcal{M}_G$, in fact even the topology of the coproduct of infinitely many topological G-modules, is not easy, but doable - see [**25**], [**11**], and [**6**].

Recall that the **sup topology** of a collection of topologies T_i on a set X is the weakest topology finer than all of the T_i. It has as a basis for its open sets the sets of the form $U_1 \cap \cdots \cap U_n$ where each U_j is open in some T_i.

Lemma 1.3.12 (Sup Topology). *If $(T_i)_{i \in I}$ is a collection of topologies on a G-module (resp. G-space) A such that A is a topological G-module (resp. G-space) under each T_i, then A remains a topological G-module (resp. G-space) under the sup topology of all the T_i.*

Proof. This follows from [**38**], but we give a direct proof. To see that A is a G-space, suppose without loss of generality that $g \in G, x \in A$, and $g \cdot x \in U = U_1 \cap \cdots \cap U_n$, with U_i open in $T_i, i = 1, \ldots, n$. Then for each i there is an open set $V_i \ni g$ in G and an open set $W_i \ni x$ in T_i such that $V_i \cdot W_i \subseteq U_i$, so $V = \bigcap V_i$ is a neighborhood of g and $W = \bigcap W_i$ is a neighborhood of x such that $V \cdot W \subseteq U$.

Similarly, to see that A is an abelian topological group under T (hence topological G-module), suppose $x_1, x_2 \in A$ and $x_1 - x_2 \in U = U_1 \cap \cdots \cap U_n$, with U_i open in T_i. For each i there are open sets $V_i \ni x_1$ and $W_i \ni x_2$ with $V_i - W_i \subset U_i$, so $V = \bigcap V_i$ is a neighborhood of x_1 and $W = \bigcap W_i$ is a neighborhood of x_2 such that $V - W \subseteq U$. □

Recall that, given a map $f : X \to Y$, where Y is a topological space, the **weak topology** on X with respect to f is the coarsest topology such that f is continuous; it has for its open sets the sets $f^{-1}(U)$ such that U is open in Y.

Lemma 1.3.13. *If $B \xrightarrow{f} A$ is a morphism of G-modules (resp. G-sets) and A is a topological G-module (resp. G-space), then the weak topology T on B with respect to f makes B a topological G-module (resp. G-space).*

Proof. This also follows from [**38**], but we give a direct proof. To see that T is a topology for a G-space, note that if $g \in G, b \in B$ with $g \cdot b \in U = f^{-1}(V)$ for some open set $V \subseteq A$ then $g \cdot f(b) \in V$ so there are open sets $V_1 \ni g$ in G and $V_2 \ni f(b)$ in A such that $V_1 \cdot V_2 \subseteq V$. This implies V_1 is a neighborhood of g and $f^{-1}(V_2)$ is a neighborhood of b such that $V_1 \cdot f^{-1}(V_2) \subseteq U$, i.e. A is a G-space.

To see that T is a topology for an abelian topological group (hence topological G-module), note that if $b_1, b_2 \in B$ with $b_1 - b_2 \in U = f^{-1}(V)$ for some open set $V \subseteq A$ then there are open sets $V_i \ni f(b_i), i = 1, 2$, such that $V_1 - V_2 \subseteq V$, and so $f^{-1}(V_i)$ is a neighborhood of $b_i, i = 1, 2$, such that $f^{-1}(V_1) - f^{-1}(V_2) \subseteq U$. □

Proposition 1.3.14. *The forgetful functor from $\mathcal{M}_G$ to G-**mod** has a left and a right adjoint.*

Proof. The proof is the same as that of Proposition 1.3.7. □

Proposition 1.3.15. *The injective objects of $\mathcal{M}_G$ are the injective G-modules with trivial topology. The projective objects of $\mathcal{M}_G$ are the projective G-modules P with the largest possible topology making P a topological G-module.*

Proof. We will only do the proof for the case of injective objects, since the proof for projective objects is dual. If A is an injective G-module with trivial topology, then it is injective in $\mathcal{M}_G$ because by Proposition 1.3.14, the functor taking a G-module to a topological G-module with trivial topology has an exact left adjoint.

Conversely, suppose A is an injective topological G-module. If A does not have the trivial topology, then consider the map $\alpha = \mathrm{id} : A \to A_0$, where $A_0 = A$ as a G-module, with trivial topology, and the map $\beta = \mathrm{id} : A \to A$. Note α is a monomorphism because it is injective (setwise). There is no map γ of topological G-modules $A_0 \to A$ such that $\gamma \circ \alpha = \beta$ because the map would have to be the identity, which is not continuous.

If A is not an injective G-module then there are maps $\alpha : B \to C$ and $\beta : B \to A$ of G-modules such that there is no map γ of G-modules with $\gamma \circ \alpha = \beta$. If we put the trivial topology on C and the topology induced on B by β, then these become maps of topological G-modules. □

The preceding proposition can be interpreted in two different ways. On the one hand, with the usual definition of "enough," it shows that there are indeed enough injectives and projectives. Any topological G-module A injects into one with the trivial topology: pick an injective G-module B such that there is an injection $A \hookrightarrow B$ in G-**mod** and put the trivial topology on B; dually for projective G-modules. However, this does us no good because we want our injection $A \hookrightarrow B$ to be a proper map, since exact sequences can only be defined for proper maps, so putting the trivial topology on B would force the topology on A to be trivial, hence only topological G-modules with trivial topologies could possibly have an injective resolution. Thus there are not enough injectives or projectives in the sense that we would want.

1.3.3.4. *Abelian Group Objects in $\mathcal{C}_G$.* Let y be the functor from $\mathcal{C}_G$ to the category of presheaves of sets on $\mathcal{M}_G$ taking a G-space X to the presheaf $\mathrm{Hom}_{\mathcal{C}_G}(-, X)$.

Proposition 1.3.16. *A presheaf on $\mathcal{C}_G$ represented by a G-space X is a presheaf of abelian groups if and only if X is a topological G-module. Any homomorphism $y(A) \to y(B)$ of presheaves of abelian groups for $A, B \in \mathcal{M}_G$ comes from a unique morphism of topological G-modules $A \to B$. That is, the category of abelian group objects in $\mathcal{C}_G$ is the category $\mathcal{M}_G$.*

Proof. It is well-known that for a category $\mathcal{C}$ with a final object pt and finite products the following two definitions of an abelian group object X in $\mathcal{C}$ are equivalent: (1) the functor $\mathrm{Hom}(-, X)$ from $\mathcal{C}$ to Set factors through **Ab**.

(2) We have the following data:

(a) A map $X \times X \xrightarrow{m} X$ in $\mathcal{C}$

(b) A map $e : pt \to X$

(c) A map $\mathrm{inv} : X \to X$

satisfying the commutative diagrams for associativity, identity, inverse, and commutativity conditions. Clearly the latter definition is equivalent to X being a topological G-module, since these conditions say that X is an abelian group; the map $m : X \times X \to X$ being a map of G-spaces says that addition is continuous and $g \cdot (x + y) = g \cdot x + g \cdot y$; and inv being a map of G-spaces says that inversion $X \to X$ is continuous.

Addition of functions is addition pointwise, i.e. for any $x \in X$, where $X \in \mathcal{C}_G$, and any two morphisms $f, g : X \to A$, where $A \in \mathcal{M}_G$, we have $(f + g)(x) = f(x) + g(x)$. This is because if $(f, g) : X \to A \times A$ is the product of f and g and $\pi_1, \pi_2 : A \times A \to A$ are the two projections, then addition on A is given by $\pi_1 + \pi_2$ and $(\pi_1 + \pi_2)(f, g) = \pi_1(f, g) + \pi_2(f, g) = f + g$ (the first equality is due to the fact that the map $\mathrm{Hom}(A \times A, A) \to \mathrm{Hom}(X, A)$ given by precomposing with (f, g) is a homomorphism of abelian groups).

Now, the Yoneda correspondence says that a map $\alpha : y(A) \to y(B)$ corresponds to the map $f = \alpha(\mathrm{id}_A) : A \to B$ of G-spaces. We just have to show that f is a homomorphism of abelian groups. Since f induces a homomorphism of abelian groups $\mathrm{Hom}(A \times A, A) \to \mathrm{Hom}(A \times A, B)$ we have $f(\pi_1 + \pi_2) = f\pi_1 + f\pi_2$. Applying this to an element $(x, y) \in A \times A$ we get $f(x + y) = f(\pi_1 + \pi_2)(x, y) = (f\pi_1 + f\pi_2)(x, y) = f(x) + f(y)$ (the last equality follows from the fact that addition of functions is pointwise, proved in the last paragraph). $\square$

CHAPTER 2

Cohomology Theories Using Grothendieck Topologies

In this chapter we show that certain cohomology theories for topological groups which are defined by using cochains ("cochain theories") can be reinterpreted as cohomologies of Grothendieck topologies (Section 2.1). The original motivation for this result is that it allows these theories to be applied to number theory as in **[17]**. However, this reinterpretation may also allow for new comparisons of these theories to Wigner's semisimplicial cohomology **[35]**; this is addressed in Section 2.2.6.

In Section 2.2 we give some examples of new cohomology theories which we feel are natural, for a topological group G using topologies on the category C_G of G-spaces. We also recall Lichtenbaum's topology **[17]**, which we denote by T_G^L, and compare these topologies to one another. We then give a comparison of the semisimplicial theory $H_{ss}^n(G, A)$ of Wigner to the cochain theories from Section 2.1.

In Section 2.3 we show that, for any topology T on C_G or $C_{G,*}$ (the category of pointed G-spaces), any short exact sequence

$$0 \to A \to B \xrightarrow{\tau} C \to 0$$

of topological G-modules such that τ has a refinement by a covering in T, gives rise to long exact sequences on cohomology.

2.1. Grothendieck Topologies and Cochain Theories

Throughout Section 2.1.1, let G be any topological group and A any topological G-module. We discuss several ways to define cohomology theories $H^n(G, A)$ that we will be reinterpreting as cohomologies of Grothendieck topologies. In Sections 2.1.2, 2.1.3, 2.1.4, and 2.1.5 we discuss several ways to reinterpret each theory in terms of various Grothendieck topologies and sheaves on these topologies. The purpose is to provide more applicability of these cochain theories.

2.1.1. The Cochain Theories. In this subsection we first define the four cochain theories we will be using. We note (using the theory in Section 3.2) that $H^1(G, A)$ is the same for three of the theories, and it is the same for all of the theories under some restrictions. Next, we show that the original definition (which uses inhomogeneous cochains) is equivalent to a similar definition using homogeneous cochains. Finally, for each theory we consider $\tilde{C}^n(G, A)$, the set of cochains which map $(e, \ldots, e)$ in G^n to $0 \in A$ and show that the cohomology of the complex $(\tilde{C}^n(G, A))_{n=0}^{\infty}$ is the same if we leave $\tilde{C}^0(G, A)$ as the original $C^0(G, A)$. Otherwise, the cohomology differs slightly, namely in the 0th and 1st levels.

2.1.1.1. *Definition of the Cochain Theories.* All of the cochain theories are defined as the cohomology of the complex of **inhomogeneous** $(C^n(G, A), \delta_n)$ (described below); $C^0(G, A)$ is always just A (thought of as the set of maps from $G^0 = pt$ to A); and the coboundary homomorphism is given by

$$\begin{aligned} \delta_n(f)(g_0, \ldots, g_n) &= g_0 f(g_1, \ldots, g_n) \\ &+ \sum_{k=1}^{n} (-1)^k f(g_0, \ldots, g_{k-1}g_k, \ldots, g_n) \\ &+ (-1)^{n+1} f(g_0, \ldots, g_{n-1}). \end{aligned} \tag{3}$$

The **continuous** cochain theory $H_c^n(G, A)$ is dicussed in great detail in Section 3.1. It is defined to be the cohomology of the complex of continuous cochains $C_c^n(G, A) = \{\text{continuous maps } f : G^n \to A\}$.

The **measurable** cochain theory $H_m^n(G, A)$ is discussed in Section 3.2. Throughout this chapter, $H_m^n(G, A)$ will be the cohomology of the complex of measurable cochains $C_m^n(G, A) = \{\text{measurable maps } f : G^n \to A\}$. Recall a subset of a topological space X is measurable if it is in the Borel σ-algebra of X, the σ-algebra[1] generated by the open subsets of X. A map $f : X \to Y$ of topological spaces is measurable if $f^{-1}(U)$ is measurable for every measurable subset $U \subseteq Y$. This is equivalent to saying $f^{-1}(U)$ is measurable for every open subset $U \subseteq Y$ since f^{-1} preserves complements, unions, and intersections. When G is locally compact and Polish and A is Polish, this is called Moore cohomology.

The **locally continuous** cochain theory $H^n_{\mathrm{lc}}(G, A)$ is the cohomology of the complex of locally continuous cochains [**34**]:

$$C^n_{\mathrm{lc}}(G, A) = \left\{ \begin{array}{c} f : G^n \to A \mid \exists \text{ open subset } U \subseteq G^n, (e, \ldots, e) \in U, \\ \text{such that } f|U \text{ is continuous} \end{array} \right\}$$

Note that the product of two locally continuous maps is locally continuous. Also, if we have a composition of two maps $f : X \to Y, g : Y \to Z$ such that f is continuous on some neighborhood U of $x \in X$ and g is continuous on some neighborhood V of $f(x) \in Y$ then $g \circ f$ is continuous on the neighborhood $f^{-1}(V) \cap U$ of x. Indeed, for any open set $W \subseteq Z$ we know $g^{-1}(W) \cap V$ is open in Y, so $(g \circ f)^{-1}(W) \cap (f^{-1}(V) \cap U) = f^{-1}(g^{-1}(W) \cap V) \cap U$ is open. In particular, precomposing or postcomposing a locally continuous map with a continuous one results in a locally continuous map.

The **locally continuous measurable** cochain theory $H^n_{\mathrm{lcm}}(G, A)$ is the cohomology of the complex of locally continuous measurable cochains [**15**]:

$$C^n_{\mathrm{lcm}}(G, A) = \left\{ \begin{array}{c} \text{measurable maps } f : G^n \to A \mid \exists \text{ open subset} \\ U \subseteq G^n, (e, \ldots, e) \in U, \text{ such that } f|U \text{ is continuous} \end{array} \right\}$$

2.1.1.2. $H^0(G, A)$ *and* $H^1(G, A)$. For each cochain theory, we have $H^0(G, A) = A^G$. By Lemma 3.2.2, any crossed homomorphism which is continuous at some point

[1] A σ-algebra of subsets of X is a collection of subsets which is closed under taking complements, countable unions, and countable intersections.

is continuous everywhere, so

$$H^1_c(G, A) \cong H^1_{\mathrm{lc}}(G, A) \cong H^1_{\mathrm{lcm}}(G, A)$$

for any G and any A. On the other hand, if G is a first countable Baire group (see Section 3.2) and A is a second countable topological G-module, then Theorem 3.2.6 implies $H^1_c(G, A) \cong H^1_m(G, A)$.

2.1.1.3. *Homogeneous vs. Inhomogeneous Cochains.* For each cochain theory $H^n_\bullet(G, A)$ defined above, we define the complex of **homogeneous cochains** by setting $C^n_{\bullet,h}(G, A)$ to be the set of G-equivariant maps $G^{n+1} \to A$ that satisfy the respective properties in the definition of the cochain theory. Here, G^{n+1} is the product of $n + 1$ copies of G, with diagonal G-action, and the coboundary homomorphism is defined by

$$d_n(f)(g_0, \ldots, g_n) = \sum_{i=0}^{n} (-1)^i f(g_0, \ldots, \hat{g}_i, \ldots, g_n) \tag{4}$$

(as usual, $\hat{g}_i$ means the g_i term is omitted).

Theorem 2.1.1. *Each of the cohomology groups $H^n_\bullet(G, A)$ is isomorphic to the cohomology of the homogeneous complex $(C^n_{\bullet,h}(G, A), d_n)$.*

Proof. We use the standard argument in group cohomology, found for example in [**30**, p. 112-3]. Explicitly, we use the isomorphism of abelian groups $C^n_{\bullet,h}(G, A) \xrightarrow{\sim} C^n_\bullet(G, A)$ given by precomposing with the continuous map

$$G^n \to G^{n+1} : (g_1, \ldots, g_n) \xrightarrow{\phi_n} (e, g_1, g_1 g_2, \ldots, g_1 \cdots g_n).$$

Note that if $f : G^{n+1} \to A$ is continuous, measurable, and/or locally continuous, then so is $f \circ \phi_n$. The inverse isomorphism $C^n_\bullet(G, A) \xrightarrow{\sim} C^n_{\bullet,h}(G, A)$ takes a map $f : G^n \to A$ to the G-equivariant map $\tilde{f}(g_0, g_1, \ldots, g_n) = g_0 \cdot f(g_0^{-1} g_1, g_1^{-1} g_2, \ldots, g_{n-1}^{-1} g_n)$. Thus, $\tilde{f}$ is the composition

$$G^{n+1} \xrightarrow{b} G \times G^n \xrightarrow{\mathrm{id} \times f} G \times A \xrightarrow{a} A$$

where a is the action of G on A and $b(g_0, \ldots, g_n) = (g_0, (g_0^{-1}g_1, \ldots, g_{n-1}^{-1}g_n))$. Note that both a and b are continuous, so if f is continuous, measurable, and/or locally continuous, the same is true for $\tilde{f}$. □

Remark. If f is measurable, G has measurable multiplication $G \times G \to G$ and inversion $G \to G$, and A has measurable G-action $G \times A \to A$, then $\tilde{f}$ is still measurable, so the argument still applies; this is used in Section 2.1.3.

2.1.1.4. *Cochains That Take the Identity to* 0. Let $C^n(G, A) = C^n_{\bullet,h}(G, A)$ be the set of homogeneous cochains for any of the cochain theories above, and for $n \geq 1$ let $\tilde{C}^n(G, A)$ be the subset of $C^n(G, A)$ consisting of maps $f : G^{n+1} \to A$ which take $(e, \ldots, e)$ to $0 \in A$. Let $\tilde{C}^0(G, A) = C^0(G, A) \cong A$. Define the coboundary operators for the two complexes as before, by equation (4) (note that the image of $\tilde{C}^0(G, A)$ does indeed land in $\tilde{C}^1(G, A)$). We will show that the cohomologies of the two complexes are the same[2]. There is a short exact sequence of complexes

$$
\begin{array}{cccccccccc}
 & 0 & & 0 & & 0 & & 0 & & \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 \to & \tilde{C}^0(G,A) & \xrightarrow{\delta} & \tilde{C}^1(G,A) & \xrightarrow{\delta} & \tilde{C}^2(G,A) & \xrightarrow{\delta} & \tilde{C}^3(G,A) & \xrightarrow{\delta} \cdots \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 \to & C^0(G,A) & \xrightarrow{\delta} & C^1(G,A) & \xrightarrow{\delta} & C^2(G,A) & \xrightarrow{\delta} & C^3(G,A) & \xrightarrow{\delta} \cdots \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & 0 & \to & A & \xrightarrow{\mathrm{id}} & A & \xrightarrow{0} & A & \xrightarrow{\mathrm{id}} \cdots \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & 0 & & 0 & & 0 & & 0 & &
\end{array}
$$

where the vertical maps $C^n(G, A) \to A$ are "evaluation at $(e, \ldots, e) \in G^{n+1}$" (for $n \geq 1$) and the vertical maps $\tilde{C}^n(G, A) \to C^n(G, A)$ are inclusions. The maps $C^n(G, A) \to A$ are surjective because, for example, for every $a \in A$ the map $f(g_0, \ldots, g_n) = g_0 \cdot a$ maps to $a \in A$. The cohomology of the bottom complex is 0, so if $\tilde{H}^n(G, A)$ is the

[2]This argument is due to Thomas Goodwillie (private communication).

cohomology of the top complex and $H^n(G,A)$ is the cohomology of the middle complex, then $\tilde{H}^n(G,A) = H^n(G,A)$ for all n.

Remark 1. The same argument shows that if $C^n(G,A)$ is the set of inhomogeneous cochains and $\tilde{C}^n(G,A)$ the set of inhomogeneous cochains that take $(e,\ldots,e) \in G^n$ to $0 \in A$ with $\tilde{C}^0(G,A) = C^0(G,A) \cong A$ and coboundary given by (3) (the image of $\tilde{C}^0(G,A)$ again lands inside $\tilde{C}^1(G,A)$), then the cohomologies of the two resulting complexes are also the same.

Remark 2. It is shown in [**15**, Corollary 1] that the $H^n_{\mathrm{lcm}}(G,-)$ form an exact connected sequence of functors with respect to the short exact sequences $0 \to A' \xrightarrow{p} A \xrightarrow{q} A'' \to 0$ in $\mathcal{M}_G^p$ which are **locally split**, i.e. there is a neighborhood U of 0 in A'' and a map $s : U \to A$ such that $q \circ s = \mathrm{id}_U$. This fact may be easier to see if we define the cohomology using $\tilde{C}^n(G,A)$, the set of locally continuous measurable maps $f : G^n \to A$ such that $f(e,\ldots,e) = 0$, with $\tilde{C}^0(G,A) = A$, as in Section 2.1.1.4.

Remark 3. If we work with homogeneous cochains and set

$$\tilde{C}^0(G,A) = \{G\text{-equivariant maps } f : G \to A \mid f(e) = 0\}$$

then $\tilde{C}^0(G,A) = 0$ and the cohomology $H^n(G,A)$ of the resulting complex changes for $n = 0,1$: $H^0(G,A) = 0$ and $H^1(G,A) = Z^1(G,A)$ is the set of crossed homomorphisms (without taking quotient by the coboundaries). A similar comment applies if working with inhomogeneous cochains. Of course, $H^1(G,A)$ does not change if the action of G on A is trivial because in this case the coboundaries are 0.

2.1.2. Continuous Cochains. Let T_G^c be the topology on the category C_G of G-spaces whose coverings are single maps $\{f : X \to Y\}$ such that there is a continuous (not necessarily G-equivariant) section $s : Y \to X$ (so $f \circ s = \mathrm{id}_Y$).

Proposition 2.1.2. *T_G^c is subcanonical.*

Proof. For any G-space Z and any covering $\{f : X \to Y\}$, the diagram

$$\mathrm{Hom}(Y,Z) \to \mathrm{Hom}(X,Z) \rightrightarrows \mathrm{Hom}(X \times_Y X, Z)$$

is exact because we know it is exact in the category of G-sets, i.e. given a map $h : X \to Z$ such that $h \circ \pi_1 = h \circ \pi_2$ ($\pi_1, \pi_2 : X \times_Y X \to X$ are the projections), there is a unique $h' : Y \to Z$ with $h' \circ f = h$. But then $h' = h' \circ f \circ s = h \circ s$ is continuous. □

Remark 1. This proposition will be used in Sections 2.1.3, 2.1.4, and 2.1.5 as well. Note that if $h : X \to Z$ is measurable and the section s is measurable (or continuous), then $h' = h \circ s$ is also measurable. If h is continuous on some neighborhood of $x \in X$, $f : (X, x) \to (Y, y)$ is a locally continuous map, and there is a locally continuous (or continuous) section $s : (Y, y) \to (X, x)$ then $h' = h \circ s$ is a composition of locally continuous maps, hence is locally continuous. Similarly, if h is locally continuous and measurable and $s(Y, y) \to (X, x)$ is locally continuous and measurable (or just continuous) then $h' = h \circ s$ is locally continuous and measurable. In fact, if s is continuous, then it is not essential that $s(y) = x$ since the composition of a locally continuous function with a continuous function is locally continuous.

Lemma 2.1.3. *$\{G \times X \xrightarrow{\text{id}} G \times X\}$ is a cofinal covering in T^c_G for any topological space X with trivial G-action.*

Proof. Given a covering $\{f : Y \to G \times X\}$, there is a continuous section $s : G \times X \to Y$. Define $t : G \times X \to Y$ by $t(g, x) = g \cdot s(e, x)$. Then $f(t(g, x)) = f(g \cdot s(e, x)) = g \cdot f(s(e, x)) = g \cdot (e, x) = (g, x)$, i.e. $f \circ t = \text{id}_{G \times X}$, so $\{G \times X \xrightarrow{\text{id}} G \times X\}$ refines $\{f : Y \to G \times X\}$. □

Remark 2. Note that t in the above lemma is the composition

$$G \times X \xrightarrow{\text{id} \times e \times \text{id}} G \times G \times X \xrightarrow{\text{id} \times s} G \times Y \xrightarrow{a} G$$

where $a : G \times Y \to G$ is the G-action and e is the constant map with value $e \in G$. Therefore, if s is measurable and/or locally continuous, so is t (this is used in Sections 2.1.3, 2.1.4, and 2.1.5).

Lemma 2.1.4. *$\{G^n \xrightarrow{\text{id}} G^n\}$ is a cofinal covering (of G^n) in T^c_G for every $n \geq 1$ (where the action of G on G^n is diagonal). In fact, for $n = 1$ this is true for any topology T on C_G.*

Proof. For $n = 1$, suppose we have a covering $\{X_i \xrightarrow{f_i} G\}$. Pick any i. Since f_i is a map of G-sets, there is an $x \in X_i$ with $f_i(x) = e \in G$. Define $f : G \to X_i$ by $f(g) = g \cdot x$; then f is continuous since X_i is a G-space, and $f_i \circ f = \mathrm{id}_G$. Now assume $n > 1$. Let $G_n = G \times G^{n-1}$, where the action of G on G_n is on the left coordinate, i.e. $g \cdot (g_1, g_2, \ldots, g_n) = (gg_1, g_2, \ldots, g_n)$. Clearly, G_n is a G-space. There is an isomorphism $\phi : G^n \to G_n$ of G-spaces defined by $\phi(g_1, \ldots, g_n) = (g_1, g_1^{-1}g_2, \ldots, g_{n-1}^{-1}g_n)$. The inverse is given by $\phi^{-1}(g_1, \ldots, g_n) = (g_1, g_1g_2, \ldots, g_1g_2 \cdots g_n)$. Now $\{G_n \xrightarrow{\mathrm{id}} G_n\}$ is a cofinal covering in T_G^c by Lemma 2.1.3, so $\{G^n \xrightarrow{\mathrm{id}} G^n\}$ is a cofinal covering as well. □

Remark 3. Note that, since the isomorphism ϕ is continuous and has continuous inverse, it is in particular locally continuous and measurable, and has locally continuous measurable inverse (this is used in Sections 2.1.3, 2.1.4, and 2.1.5). Furthermore, it can be written as a composition of products of the G-operations and projections onto certain coordinates, and if these are measurable instead of being continuous, ϕ is still measurable, and so is ϕ^{-1} (this is used in Section 2.1.3).

Lemma 2.1.5. *If $\{X \xrightarrow{\mathrm{id}} X\}$ is a cofinal covering of X in a Grothendieck topology T, then the functor $F \mapsto F(X)$ is an exact functor from $\mathcal{S}(T)$ to* **Ab**.

Proof. The functor $\Gamma(-, X)$ is always left-exact, so we just have to show that if we have an epimorphism $F_1 \xrightarrow{\alpha} F_2$ of sheaves on T then $\alpha(X) : F_1(X) \to F_2(X)$ is surjective. For every section $s \in F_2(X)$ there exist a covering $\{U_i \to X\}$ and elements $s_i \in F_1(U_i)$ for all i such that $s|U_i = \alpha(U_i)(s_i)$ for all i by Lemma 1.1.5. But $\{X \xrightarrow{\mathrm{id}} X\}$ is cofinal, so it factors through some U_i, which means we have $\alpha(X)(s_i|X) = (\alpha(U_i)(s_i))|X = (s|U_i)|X = s$, i.e. $\alpha(X)$ is surjective. □

Remark 4. This lemma implies that if the coverings in a topology T on a category C are just families of single maps $\{Y \xrightarrow{f} X\}$ such that there is a section s of f (i.e. $f \circ s = \mathrm{id}_X$) in C, then for every object X in C, the covering $\{X \xrightarrow{\mathrm{id}} X\}$ is cofinal, so for any sheaf F on T the functor $F \mapsto F(X)$ is exact, which means $H^n(T, X, F) = 0$

for all $n \geq 1$, all X, and all F, i.e. T has no cohomology! In fact, such a topology is equivalent to the trivial topology whose coverings are just isomorphisms.

THEOREM 2.1.6. *For any topological G-module A, the cohomology $H^n(T_G^c, pt, \tilde{A})$ is isomorphic to the continuous cochain cohomology $H_c^n(G, A)$.*

PROOF. By Lemmas 2.1.4 and 2.1.5, for any sheaf F on T_G^c we have $H^n(T_G^c, G^n, F) = 0$ since the functor $F \mapsto F(G^n)$ is exact. By Lemma 1.1.7, $H^n(T_G^c, pt, F) \cong H^n(\{G \to pt\}, F)$. Now, if $F = \tilde{A}$ then $H^n(\{G \to pt\}, \tilde{A})$ is precisely the cohomology of the homogeneous complex of continuous cochains, i.e. $H^n(\{G \to pt\}, \tilde{A}) = H_{c,h}^n(G, A) \cong H_c^n(G, A)$ by Theorem 2.1.1. □

2.1.3. Measurable Cochains. Consider the category C_G^m of G-spaces and measurable G-equivariant maps. In C_G^m, if we have two morphisms $f : X \to Y$ and $g : Z \to Y$ then $X \times_Y Z = \{(x, z) \in X \times Z \mid f(x) = g(z)\}$ with the subspace topology is the fibered product in this category. First note that the projections $\pi_1 : X \times_Y Z \to X$ and $\pi_2 : X \times_Y Z \to Z$ are the compositions of the (continuous) inclusion $X \times_Y Z \hookrightarrow X \times Z$ and the corresponding projections from $X \times Z$, which are continuous, hence π_1 and π_2 are continuous, and in particular measurable. As in C_G, π_1 and π_2 are G-equivariant. Now if we have two morphisms $\alpha : W \to X$ and $\beta : W \to Z$ in C_G^m such that $f \circ \alpha = g \circ \beta$ then there is a map $(\alpha, \beta) : W \to X \times Z$ which factors as $W \xrightarrow{\gamma} X \times_Y Z \hookrightarrow X \times Z$. The map (α, β) is measurable since if U and V are open sets in X and Z, respectively, then $(\alpha, \beta)^{-1}(U \times V) = \alpha^{-1}(U) \cap \beta^{-1}(V)$ is the intersection of measurable sets, hence is measurable. But this implies that γ is measurable, because if U is an open subset of $X \times_Y Z$ then $U = U' \cap X \times_Y Z$ for some open set $U' \subseteq X \times Z$, and $\gamma^{-1}(U) = (\alpha, \beta)^{-1}(U')$ is measurable. Finally, the map γ is the unique map of G-sets such that $\pi_1 \circ \gamma = \alpha$ and $\pi_2 \circ \gamma = \beta$ by consideration of the sets involved.

Define a topology T_G^m on C_G^m where the coverings are families consisting of single morphisms of the form $\{X \xrightarrow{f} Y\}$ such that f has a measurable (not necessarily G-equivariant) section $s : Y \to X$ (so $f \circ s = \mathrm{id}_Y$). It is easy to check that this does actually give a topology: isomorphisms have a measurable section, a composition of measurable sections is a measurable section, and if we have two measurable functions $f : X \to Z, g : Y \to Z$ and f has a measurable section then $\tilde{f} : X \times_Z Y \to Y$ has the measurable section $y \mapsto (s(g(y)), y)$. In fact, this topology is subcanonical by Remark 1 in Section 2.1.2.

Theorem 2.1.7. *For any topological G-module A, the cohomology $H^n(T^m_G, pt, \tilde{A})$ is isomorphic to the Moore cohomology $H^n_m(G, A)$.*

Proof. The proof is the same as that of Theorem 2.1.6. □

Proposition 2.1.8. *Consider the morphism of topologies $g : T^c_G \to T^m_G$ taking a G-space to itself (note that continuous maps are measurable). The functor g_* is exact.*

Proof. Using Lemma 1.1.6, it is enough to show any covering $\{f : X \to Y\}$ in T^m_G has a refinement by a covering in T^c_G; indeed, it has a refinement by $\{a : G \times Y_{\text{triv}} \to Y\}$, where Y_{triv} is Y as a topological space but with trivial G-action and $a(g, y) = g \cdot y$ (this is just the action of G on Y, which is continuous because Y is a G-space, after all). Making Y have the trivial G-action instead of the original makes the map a G-equivariant. And $\{a : G \times Y_{\text{triv}} \to Y\}$ refines $\{f : X \to Y\}$ since we can define a measurable G-map $h : G \times Y_{\text{triv}} \to X$ by $h(g, y) = g \cdot s(y)$, where s is a measurable section of f; we can easily check $f \circ h = a$. □

Corollary 2.1.9. *Let A be a topological G-module. Then* $\text{Hom}_{C^m_G}(-, A)$ *is a sheaf on T^c_G, and $H^n(T^c_G, pt, \text{Hom}_{C^m_G}(-, A)) \cong H^n_m(G, A)$ for all n.*

Proof. $\text{Hom}_{C^m_G}(-, A)$ is precisely $g_*\tilde{A}$ in the notation of Proposition 2.1.8, so $H^n(T^c_G, pt, \text{Hom}_{C^m_G}(-, A)) \cong H^n(T^m_G, pt, \tilde{A}) \cong H^n_m(G, A)$. □

Remark 1. Actually, if we redefined the measurable cohomology groups $H^n_m(G, A)$ in a broader setting, Theorem 2.1.7 would be true for any abelian group object A of C^m_G, i.e. any G-space A with a commutative measurable "addition" map $A \times A \to A$, an "identity" map $0 \to A$, and a measurable "additive inversion" map $A \to A$ satisfying the usual axioms. We will do this presently, in even more generality.

The following construction is worth mentioning because it also gives the measurable cochain theory, but in what seems to be the fullest generality. The construction requires some definitions from Mackey's work on Borel spaces [**18**], so we review the necessary definitions and facts here, putting them into the context of category theory. We define a category $\mathcal{B}$ as follows. The objects are sets X with a **Borel structure**: a σ-algebra of subsets of X. We call these subsets the **Borel subsets** of X. The objects of $\mathcal{B}$ are called **Borel spaces**. If the Borel structure on X is the σ-algebra generated by the open sets in some topology on X then we say the Borel structure is **topological.** A morphism $f : X \to Y$ of Borel spaces is a **Borel map** of sets, i.e. $f^{-1}(U)$ is Borel for every Borel set U in Y. Of course, if the Borel structures on X and Y are topological, then any continuous map $f : X \to Y$ is Borel.

A subset U of the Borel space X is a Borel **subspace** if U has the σ-algebra consisting of all sets $V \cap U$ such that V is Borel in X. A map $f : Y \to U$ is Borel if and only if the induced map $f : Y \to X$ is Borel. If the Borel structure on X is topological, then the Borel structure for U is generated by the subspace topology.

Products of Borel spaces exist: for an arbitrary family $\{X_i\}_{i \in I}$ of Borel spaces X_i, the categorical product is $X = \prod_{i \in I} X_i$ with the σ-algebra of subsets generated by the sets $\pi_i^{-1}(U)$ such that U is a Borel subset of X_i, where $\pi_i : X \to X_i$ is the usual projection. If the Borel structures on X_i are topological, then the Borel structure on X is generated by the product topology. This implies arbitrary projective limits, and in particular fibered products, exist in this category as well.

If $\sim$ is any equivalence relation on the Borel space X, $Y = X/\sim$ inherits a structure of a Borel space: $U \subseteq Y$ is defined to be a Borel subset if $q^{-1}(U)$ is Borel in X, where $q : X \to Y$ is the **quotient map** taking $x \in X$ to its equivalence class. This is the largest Borel structure such that q is Borel and a map $f : Y \to Z$ for some other Borel space Z is Borel if and only if $f \circ q$ is. Unfortunately, if the Borel structure on X comes from a topology, then the resulting Borel structure on Y does not necessarily come from the quotient topology.

Example. Let $X = \operatorname{Spec} \mathbb{Z} = \{0\} \cup \{p_i\}_{i=1}^{\infty}$ (where p_i stands for the i-th prime, though this example has nothing to do with number theory or algebraic geometry). The topology on X is such that a nonempty set $U \subseteq X$ is open if and only if $0 \in U$ and U is missing only finitely many points. Consider the equivalence relation given by $0 \sim 0$ and $p_i \sim p_j$ for all i, j. Then $Y = X/\sim = \{0, p\}$ with the trivial topology, hence the Borel σ-algebra on Y is the trivial one. But if $q : X \to Y$ is the quotient map, then $q^{-1}(\{0\}) = \{0\} = \cap_i (X \setminus \{p_i\})$ is the intersection of countably many open sets, hence is in the Borel σ-algebra for X, so $\{0\}$ is in the quotient Borel structure for Y. Thus, the quotient topology does not induce the quotient Borel structure for Y.

Let G be a **Borel group**: a group which is also a Borel space such that "multiplication" $G \times G \to G$ and "inversion" $G \to G$ are Borel functions, i.e. a group object in $\mathcal{B}$ (a group object technically requires that the "identity" map $pt \to G$ be Borel but this is automatically true). Let $\mathcal{B}_G$ be the category whose objects are Borel spaces with a Borel G-action $G \times X \to X$ and morphisms are Borel G-maps. Note that fibered products exist by the same proof as for the category $\mathcal{B}$, or by the proof for the category $\mathcal{C}_G^m$. Let $T_{G,B}$ be the topology on $\mathcal{B}_G$ whose coverings consist of single morphisms $X \xrightarrow{f} Y$ which have a Borel global section. $T_{G,B}$ is indeed a subcanonical topology, by Remark 1 in Section 2.1.2.

For a Borel group G, we call a G-module A a **Borel G-module** if its G-action $G \times A \to A$, addition $A \times A \to A$, and additive inversion $A \to A$ are all Borel maps. We define $C_B^n(G, A)$ to be the set of Borel maps from G^n to A, where $C_B^0(G, A) = A$ is the set of constant maps, and use the inhomogeneous coboundary operator (equation (3)) to define a complex, whose n-th cohomology we denote by $H_B^n(G, A)$. Of course, if G is a topological group and A is a topological G-module then $H_B^n(G, A) = H_m^n(G, A)$.

Theorem 2.1.10. *$H^n(T_{G,B}, pt, \tilde{A}) \cong H_B^n(G, A)$ for all $n \geq 0$.*

Proof. The proof is the same as before. The proofs of Lemmas 2.1.3 and 2.1.4 and Theorem 2.1.1 rely on compositions of addition/subtraction on A, the G-action on A, and multiplication in G, all of which are Borel functions, and the composition of Borel functions is Borel. See Remarks 2 and 3 in Section 2.1.2 and Remark 1 in Section 2.1.1.3. □

2.1.4. Locally Continuous Cochains. In this section we introduce a Grothendieck topology T_G^{lc} whose cohomology $H^n(T_G^{\mathrm{lc}}, pt, \mathrm{Map}_{G,\mathrm{lc}}(-, A))$ is isomorphic to that of the locally continuous cochain theory $H^n_{\mathrm{lc}}(G, A)$ discussed in **[34]** and defined in Section 2.1.1. Here, for a pointed G-space (X, x), $\mathrm{Map}_{G,\mathrm{lc}}((X, x), A)$ is the set of G-equivariant maps $X \to A$ (not necessarily mapping x to 0) that are continuous on some neighborhood of x.

Let C_G^{lc} be the category of pointed G-spaces (X, x) and **locally continuous** morphisms, i.e. G-maps $f : (X, x) \to (Y, y)$ such that $f(x) = y$ and there is a neighborhood U of x such that $f|U$ is continuous. Note C_G^{lc} is actually a category, since a composition of two locally continuous morphisms is locally continuous, by the argument in Section 2.1.1.

Fibered products exist in C_G^{lc}: if we have two morphisms $f : (X, x) \to (Y, y)$ and $g : (Z, z) \to (Y, y)$ then the fibered product is $(X \times_Y Z, (x, z))$. The proof is basically the same as for G-spaces. The projections $X \times Z \to X$ and $X \times Z \to Z$ are continuous, hence so are the maps $X \times_Y Z \to X$ and $X \times_Y Z \to Z$. The only other thing to check is that if we have two maps $\alpha : (W, w) \to (X, x)$ and $\beta : (W, w) \to (Z, z)$ with $f \circ \alpha = g \circ \beta$ then the induced map $\gamma : (W, w) \to (X \times_Y Z, (x, z))$ is continuous in a neighborhood of w. This is easy: if α is continuous in a neighborhood U_α of w and β is continuous in a neighborhood U_β of w then the map $(\alpha, \beta) : W \to X \times Z$ is continuous on $U_\alpha \cap U_\beta$, and hence so is γ.

Define the topology T_G^{lc} on C_G^{lc} by saying the coverings are single-morphism families $\{(X, x) \xrightarrow{f} (Y, y)\}$ such that f is surjective and there is a local section s of f. This is equivalent to saying there is a morphism $s : (Y, y) \to (X, x)$ in C_G^{lc} with

$f \circ s = \mathrm{id}_Y$. It is easy to check that this actually gives a subcanonical topology; the proof is the same as in Section 2.1.2 (the condition $s(y) = x$ is necessary for this).

Theorem 2.1.11. *For any topological G-module A, $\mathrm{Map}_{G,\mathrm{lc}}(-, A)$ is a sheaf on T_G^{lc} and the cohomology $H^n(T_G^{\mathrm{lc}}, pt, \mathrm{Map}_{G,\mathrm{lc}}(-, A))$ is isomorphic to the locally continuous cohomology $H^n_{\mathrm{lc}}(G, A)$.*

Proof. First, $\mathrm{Map}_{G,\mathrm{lc}}(-, A)$ is a sheaf by Remark 1 of Section 2.1.2. By the proof of Lemma 2.1.4, $\{G^n \xrightarrow{\mathrm{id}} G^n\}$ is a cofinal covering. Thus for any sheaf F on T_G^{lc}, $H^n(T_G^{\mathrm{lc}}, G^n, F) = 0$ for $n \geq 1$ by Lemma 2.1.5. Therefore, $H^n(T_G^{\mathrm{lc}}, pt, \mathrm{Map}_{G,\mathrm{lc}}(-, A)) \cong H^n(\{G \to pt\}, \mathrm{Map}_{G,\mathrm{lc}}(-, A))$ by Lemma 1.1.7. But the latter is precisely the cohomology of the homogeneous complex of locally continuous cochains $H^n_{\mathrm{lc},h}(G, A)$, which is isomorphic to $H^n_{\mathrm{lc}}(G, A)$ by Theorem 2.1.1. □

Remark 1. Note that the sheaf represented by $(A, 0)$ in T_G^{lc} does not quite give the same cohomology as $\mathrm{Map}_{G,\mathrm{lc}}(-, A)$. In fact, $H^n(T_G^{\mathrm{lc}}, pt, \mathrm{Hom}_{C_G^{\mathrm{lc}}}(-, A)) \cong H^n(\{G \to pt\}, \mathrm{Hom}_{C_G^{\mathrm{lc}}}(-, A))$, and the latter is the cohomology of the complex $(C^n(G, A) = \mathrm{Hom}_{C_G^{\mathrm{lc}}}(G^{n+1}, A), d_n)$ consisting of locally continuous homogeneous cochains which take $(e, \ldots, e) \in G^{n+1}$ to $0 \in A$. This cohomology is discussed in Remark 3 of Section 2.1.1.4.

We can also obtain the same cohomology via T_G^c:

Theorem 2.1.12. *For any topological G-module A, $\mathrm{Map}_{G,\mathrm{lc}}(-, A)$ is a sheaf on T_G^c and the cohomology $H^n(T_G^c, pt, \mathrm{Map}_{G,\mathrm{lc}}(-, A))$ is isomorphic to the locally continuous cohomology $H^n_{\mathrm{lc}}(G, A)$.*

Proof. The proof is the same as that of Theorem 2.1.11. □

We can also consider instead the category $C_{G,*}$ of pointed G-spaces with G-equivariant continuous maps $f : (X, x) \to (Y, y)$ with $f(x) = y$ and the topology $T_{G,*}^c$ on $C_{G,*}$ whose coverings are single maps $\{f : (X, x) \to (Y, y)\}$ such that there is a continuous (not necessarily G-equivariant) section $s : (Y, y) \to (X, x)$ (i.e. $f \circ s = \mathrm{id}_Y$

and $s(y) = x$). There is a morphism of topologies $\alpha : T^c_{G,*} \to T^c_G$ given by "forgetting" the special point in a pointed G-space.

PROPOSITION 2.1.13. *For any sheaf F on T^c_G and any pointed G-space (X, x) we have $H^n(T^c_{G,*}, (X, x), \alpha_* F) \cong H^n(T^c_G, X, F)$.*

PROOF. This is a special case of Theorem 2.2.26 below. □

COROLLARY 2.1.14. *For any topological G-module A, $\mathrm{Hom}_{C_G}(\alpha(-), A)$ is a sheaf on $T^c_{G,*}$ and $H^n(T^c_{G,*}, pt, \mathrm{Hom}_{C_G}(\alpha(-), A)) \cong H^n(T^c_G, pt, \tilde{A}) \cong H^n_c(G, A)$ for all n.*

COROLLARY 2.1.15. *For any topological G-module A, $\mathrm{Map}_{G,\mathrm{lc}}(\alpha(-), A)$ is a sheaf on $T^c_{G,*}$ and $H^n(T^c_{G,*}, pt, \mathrm{Map}_{G,\mathrm{lc}}(\alpha(-), A)) \cong H^n(T^c_G, pt, \mathrm{Map}_{G,\mathrm{lc}}(\alpha(-), A)) \cong H^n_{\mathrm{lc}}(G, A)$ for all n.*

There is also a morphism of topologies $\beta : T^c_{G,*} \to T^{lc}_G$ since continuous maps are locally continuous.

PROPOSITION 2.1.16. *For any sheaf F on T^{lc}_G and any pointed G-space (X, x) we have $H^n(T^c_{G,*}, (X, x), \beta_* F) \cong H^n(T^{lc}_G, (X, x), F)$.*

PROOF. By Lemma 1.1.6, the proposition follows from the fact that for any covering $\{f : (X, x) \to (Y, y)\}$ in T^{lc}_G there is a refinement by a covering $\{h : (G \times Y_{\mathrm{triv}}, (e, y)) \to (Y, y)\}$ in $T^c_{G,*}$ just as in the proof of Proposition 2.1.8. □

COROLLARY 2.1.17. *For any topological G-module A, $\mathrm{Hom}_{C^{lc}_G}(-, A)$ is a sheaf on $T^c_{G,*}$, and $H^n(T^c_{G,*}, pt, \mathrm{Hom}_{C^{lc}_G}(-, A)) \cong H^n(T^{lc}_G, pt, \tilde{A})$.*

2.1.5. Locally Continuous Measurable Cochains. In this section, we will define a Grothendieck topology T^{lcm}_G where $\mathrm{Map}_{G,\mathrm{lcm}}(-, A)$ is a sheaf and

$$H^n(T^{\mathrm{lcm}}_G, pt, \mathrm{Map}_{G,\mathrm{lcm}}(-, A)) \cong H^n_{\mathrm{lcm}}(G, A).$$

Here, for a pointed G-space (X, x), $\mathrm{Map}_{G,\mathrm{lcm}}((X, x), A)$ is the set of G-equivariant measurable maps $X \to A$ that are continuous in a neighborhood of x.

We consider the category C_G^{lcm} of pointed G-spaces (X, x) and maps $f : (X, x) \to (Y, y)$ such that f is G-equivariant, measurable everywhere, $f(x) = y$, and there is a neighborhood U of x such that $f|U$ is continuous. In C_G^{lcm}, if we have two morphisms $f : (X, x) \to (Y, y)$ and $g : (Z, z) \to (Y, y)$ then the fibered product is $(X \times_Y Z, (x, z))$, by the proofs of the same fact in the categories C_G^{lc} (Section 2.1.4) and C_G^m (Section 2.1.3).

Define a topology T_G^{lcm} on C_G^{lcm} where the coverings are families of the form $\{(X, x) \xrightarrow{f} (Y, y)\}$ such that there exists a section $s : (Y, y) \to (X, x)$ such that s is measurable everywhere and continuous in a neighborhood of x, and $f \circ s = \mathrm{id}_Y$. This actually gives a subcanonical topology by the remark following Proposition 2.1.2.

Theorem 2.1.18. *For any topological G-module A, $\mathrm{Map}_{G,\mathrm{lcm}}(-, A)$ is a sheaf on T_G^{lcm} and $H^n(T_G^{\mathrm{lcm}}, pt, \mathrm{Map}_{G,\mathrm{lcm}}(-, A)) \cong H^n_{\mathrm{lcm}}(G, A)$ for all n.*

Proof. The proof is the same as that of Theorem 2.1.11. □

Remark 1. Again, the cohomology of the sheaf represented by $(A, 0)$ on T_G^{lcm} is not quite the same as that of $\mathrm{Map}_{G,\mathrm{lcm}}(-, A)$, and the difference is described in Remark 3 of Section 2.1.1.4.

Theorem 2.1.19. *For any topological G-module A, $\mathrm{Map}_{G,\mathrm{lcm}}(-, A)$ is a sheaf on T_G^c and the cohomology $H^n(T_G^c, pt, \mathrm{Map}_{G,\mathrm{lcm}}(-, A))$ is isomorphic to the locally continuous cohomology $H^n_{\mathrm{lcm}}(G, A)$.*

Proof. The proof is the same as that of Theorem 2.1.11. □

Note there is a morphism of topologies $\gamma : T_{G,*}^c \to T_G^{\mathrm{lcm}}$ since continuous maps are locally continuous and measurable.

Proposition 2.1.20. *For any sheaf F on T_G^{lcm} and any pointed G-space (X, x) we have $H^n(T_{G,*}^c, (X, x), \gamma_* F) \cong H^n(T_G^{\mathrm{lcm}}, (X, x), F)$.*

Proof. The proof is the same as that of Proposition 2.1.16. □

Corollary 2.1.21. *For any topological G-module A,* $\mathrm{Hom}_{C_G^{\mathrm{lcm}}}(-,A)$ *is a sheaf on* $T^c_{G,*}$*, and* $H^n(T^c_{G,*}, pt, \mathrm{Hom}_{C_G^{\mathrm{lcm}}}(-,A)) \cong H^n(T_G^{\mathrm{lcm}}, pt, \tilde{A})$*.*

2.1.6. Remarks On These Topologies.

Proposition 2.1.22. *In any Grothendieck topology T where the coverings are single-morphism families* $\{X \to Y\}$*, for any abelian group A, the constant presheaf A is a sheaf.*

Proof. Let P be the constant presheaf. Then for any X in $\mathrm{Cat}(T)$,

$$\begin{aligned} P^\dagger(X) &= \varinjlim_{\{Y\to X\}} \ker(P(Y) \rightrightarrows P(Y \times_X Y)) \\ &= \varinjlim_{\{Y\to X\}} \ker(A \rightrightarrows A) \\ &= \varinjlim_{\{Y\to X\}} A \\ &= A \end{aligned}$$

It follows that for any X, $P^{\dagger\dagger}(X) = A$, and $P^{\dagger\dagger}$ is the constant sheaf. □

Proposition 2.1.23. *The cohomology of the constant sheaf* $\mathbb{Z}$ *for the topologies in this section is given by* $H^0(pt, \mathbb{Z}) = \mathbb{Z}$ *and* $H^n(pt, \mathbb{Z}) = 0$ *for* $n \geq 1$*.*

Proof. As we have seen, if F is the constant sheaf $\mathbb{Z}$, then

$$H^n(pt, F) = H^n(\{G \to pt\}, F)$$

is the cohomology of the complex

$$F(G) \xrightarrow{\mathrm{id}-\mathrm{id}} F(G^2) \xrightarrow{\mathrm{id}-\mathrm{id}+\mathrm{id}} F(G^3) \to \cdots$$

where $F(G^n) = \mathbb{Z}$ for each n, because the constant presheaf $\mathbb{Z}$ is a sheaf. The cohomology of this complex is precisely what is stated in the Proposition. □

Remark. In each of the topologies in this section, instead of using covers that are single maps $\{X \to Y\}$ satisfying some property (*), we could have said that a family $\{X_i \to Y\}_{i\in I}$ is a covering iff there is an index i such that the map $X_i \to Y$ satisfies property (*). In fact, such a topology would be the saturation of the topologies given here. This may be necessary for applications when combining these topologies for various groups G as in [**17**].

2.2. Topologies on the Category of G-Spaces

2.2.1. Definitions and Basic Properties. In this section, we work with several topologies on the category C_G of G-spaces that we feel may be suitable for a viable cohomology theory for G.

2.2.1.1. *The Canonical Topology T_G^{can}.* We denote the canonical topology on C_G by T_G^{can}. A collection $\mathcal{U} = \{U_i \xrightarrow{f_i} U\}_{i \in I}$ is a covering in T_G^{can} if and only if all representable presheaves satisfy the sheaf axiom for $\mathcal{U}$, i.e. for every G-space X the sequence

$$\text{Hom}(U, X) \to \prod \text{Hom}(U_i, X) \rightrightarrows \prod \text{Hom}(U_i \times_U U_j, X)$$

is exact, and the same is true under any base change. The exactness of this diagram is equivalent to the exactness of

$$\coprod U_i \times_U U_j \rightrightarrows \coprod U_i \to U$$

which says that the map $\coprod U_i \to U$ is the coequalizer of $\coprod U_i \times_U U_j \rightrightarrows \coprod U_i$. Sometimes, the terminology "regular epimorphism" is used to describe a map which is the coequalizer of two arrows, so a covering in our topology consists of a universal regular epimorphism, or a universal regular epimorphic family (if there is more than one U_i). For brevity's sake, we will call such coverings **canonical coverings**.

It is easy to see that, in fact, a morphism is a regular epimorphism if and only if it is a quotient map. Thus $\{U_i \xrightarrow{f_i} U\}$ is a covering in the canonical topology if and only if $\coprod U_i \to U$ is a universal quotient map, which means (1) $U = \bigcup f_i(U_i)$, (2) a subset $V \subseteq U$ is open if and only if $f_i^{-1}(V)$ is open for all i, and (3) these properties remain true under any base change. In particular, $\{U_i \xrightarrow{f_i} U\}$ is a covering in T_G^{can} iff $\coprod f_i : \coprod U_i \to U\}$ is a covering in T_G^{can}.

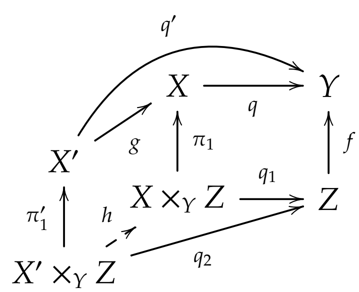
Example. There are quotient maps which are not universal, hence not coverings in T_G^{can}. All spaces will have trivial G-action. Let $X = \{a, b, b', c\}$ where the only nontrivial open set is $\{a, b\}$, $Y = \{a, b, c\}$ with trivial topology, and $Z = \{a, c\}$ with trivial topology. The map $q : X \to Y$ defined by $q(a) = a, q(b) = q(b') = b, q(c) = c$ is a quotient map (note it is not an open map - else q would be a universal quotient map). Let $f : Z \hookrightarrow Y$ be the inclusion. Then the projection π_2 from $W = X \times_Y Z$ to Z is not a quotient map because the set $U = \{(a, a)\} \subseteq X \times_Y Z$ is open ($U = (\{a, b\} \times Z) \cap W$) but $U = \pi_2^{-1}(\{a\})$ and $\{a\} \subseteq Z$ is not open.

Proposition 2.2.1. *The category of G-spaces is not a Grothendieck topos.*

Proof. A Grothendieck topos is regular **[3]**, meaning any pullback of a regular epimorphism is a regular epimorphism, and as we noted above, regular epimorphisms are just the quotient maps. □

Proposition 2.2.2. *If a map $q : X \to Y$ has a refinement by a universal quotient map, then q is itself a universal quotient map.*

Proof.



We will use the notation in the diagram above. First note that if a map $q : X \to Y$ has a refinement by a quotient map $q' : X' \to Y$ then q is a quotient map. This is because if U is a subset of Y such that $q^{-1}(U)$ is open then $g^{-1}(q^{-1}(U)) = (q')^{-1}(U)$ is open, so U is open. Now suppose q' is a universal quotient map. Then for any map $f : Z \to Y$ the induced map $q_2 : X' \times_Y Z \to Z$ is a quotient map which refines the induced map $q_1 : X \times_Y Z \to Z$, so q_1 is a quotient map. Therefore, q is a universal quotient map. □

Corollary 2.2.3. *T^{can}_G is saturated.*

Proof. Suppose a covering $\{f_i : X_i \to X\}$ has a refinement by a covering $\{q_i : Y_i \to X\}$ in T^{can}_G. Then $\{\coprod f_i : \coprod X_i \to X\}$ has a refinement by the covering $\{\coprod q_i : \coprod Y_i \to X\}$ in T^{can}_G, which is a universal quotient map, hence $\coprod f_i$ is a universal quotient map, which implies $\{f_i : X_i \to X\}$ is a covering in T^{can}_G. □

2.2.1.2. *The topology T^s_G.*

Definition 2.2.4. *Let T^s_G be the topology whose coverings are* **epimorphic (or surjective) families** $\{U_i \to U\}$ *(i.e. the map* $\operatorname{Hom}(U, V) \to \prod \operatorname{Hom}(U_i, V)$ *is injective for every V).*

Proposition 2.2.5. *A family $\{U_i \to U\}$ is epimorphic iff $U = \bigcup \operatorname{Im}(U_i)$. The topology T^s_G is strictly coarser than the canonical topology.*

Proof. First we show that not every representable presheaf is a sheaf. Let $U = \{a, b\}$ with trivial G-action and trivial topology, and let $U' = \{a, b\}$ with trivial G-action and discrete topology. Then $\{\text{id} : U' \to U\}$ is a covering in this topology. But this covering has no refinement by a canonical covering $\{U_i \to U'\}$ since T^{can}_G is saturated (Corollary 2.2.3) and $\text{id} : U' \to U$ is not a quotient map. Now $\operatorname{Hom}(-, U')$ is not a sheaf since it does not satisfy the covering $\{\text{id} : U' \to U\}$.

Now suppose $\{U_i \to U\}$ is a family of morphisms and there is an element $u \in U$ which is not in the image of any U_i. Then the whole orbit O of u is not in $\bigcup \operatorname{Im}(U_i)$. Let $V = \{a, b\}$ with trivial G-action and trivial topology. Define $f : U \to V$ by taking every element to a and $g : U \to V$ by taking every element in $U \setminus O$ to a and every element of O to b (these maps are clearly maps of G-sets, and any map into a space with a trivial topology is continuous). The images of f and g are the same in $\prod \operatorname{Hom}(U_i, V)$, but these maps are clearly not the same, so this family is not surjective. The converse, that if $U = \bigcup \operatorname{Im}(U_i)$ then $\{U_i \to U\}$ is a epimorphic family, is obvious by the usual set-theoretic argument. □

This proposition provides another explanation for why C_G is not a Grothendieck topos: in a Grothendieck topos, every epimorphic family is regular [**3**, Ch. 6, Theorem 8.13 and Exercise 6.8 (GEN)(e)].

PROPOSITION 2.2.6. *The category of sheaves of sets $\mathcal{S}'(T^s_G)$ on T^s_G is equivalent to G-**set** , hence $\mathcal{S}(T^s_G)$ is equivalent to G-**mod** . In fact, any sheaf is representable by a G-space with the trivial topology.*

PROOF. One can use the usual proof that $\mathcal{S}'(T_G)$ is equivalent to G-**set** , found for example in [**31**, Proposition I.1.3.2.1]. The quasi-inverse functors are $X \mapsto \mathrm{Hom}_{\mathcal{G}}(-, X_t)$, where X_t is X with the trivial topology (note $\mathrm{Hom}_{C_G}(Y, X_t)$ is just the set of G-equivariant maps $Y \to X$), and $F \mapsto F(G)$ (where G has its given topology and $F(G)$ has the usual G-action $g \cdot s = F(\cdot g)(s)$ for $s \in F(G)$). The only item that needs extra consideration is that the proof uses maps such as $G \to X : g \mapsto g \cdot x$ for some $x \in X$, and these maps are continuous because X is a G-space. □

2.2.1.3. *The topology T^o_G.*

DEFINITION 2.2.7. *A covering $\{U_i \xrightarrow{f_i} U\}$ such that $U = \bigcup f_i(U_i)$ and each f_i is open is called an* **open covering**. *Let T^o_G be the topology on C_G whose coverings are the open coverings.*

PROPOSITION 2.2.8. *T^o_G is a subcanonical topology, and a covering $\{U_i \xrightarrow{f_i} U\}$ is open if and only if $\{\coprod U_i \xrightarrow{\coprod f_i} U\}$ is an open covering.*

PROOF. A homeomorphism is open, and the composite of two open maps is open, so to show T^o_G is a topology we just have to check that open maps are stable under pullbacks (surjectivity is automatically preserved under pullbacks). Suppose we have two continuous maps $p : U \to X, q : V \to X$ with p open. An open set in a basis for the topology of $U \times_X V$ is of the form $W = (U' \times V') \cap U \times_X V = \{(u \in U', v \in V') \mid p(u) = q(v)\}$, where $U' \subseteq U, V' \subseteq V$ are open subsets. Then the projection $\pi_2 : U \times_X V \to V$ of W is $\{v \in V' \mid \exists u \in U', p(u) = q(v)\} = q^{-1}(p(U')) \cap V'$, which is open since p is open, so π_2 is open, as desired.

It is easy to check that a family $\{U_i \xrightarrow{f_i} U\}$ is an open covering if and only if $\{\coprod U_i \to U\}$ is a surjective open map. A surjective open map is always a universal quotient map, so the topology is subcanonical. □

Example. A map $f : X \to Y$ can have a refinement by an open surjective map without being open. Let $X = \{a, b, a'\}$ where $\{a'\}$ is the only nontrivial open subset, $Y = \{a, b\}$ with trivial topology. Define $f : X \to Y$ by $f(a) = f(a') = a$, $f(b) = b$. Then f has a refinement by id_X, but f is not open. Thus T_G^o *is not saturated.*

2.2.1.4. *The Lichtenbaum Topology* T_G^L.

Definition 2.2.9. *A* **Lichtenbaum covering** *is a family of morphisms* $\{X_i \xrightarrow{f_i} X\}$ *in* C_G *such that for all* $x \in X$ *there is a neighborhood* U *of* x*, an index* i*, and a continuous (not necessarily G-equivariant) map* $s : U \to X_i$ *with* $f_i \circ s = \mathrm{id}_U$.

The **Lichtenbaum topology** [**17**, p. 659] T_G^L *on* C_G *is the one whose coverings are the Lichtenbaum coverings.*

Proposition 2.2.10. T_G^L *is a subcanonical topology, and a family* $\{X_i \xrightarrow{f_i} X\}$ *is a Lichtenbaum covering if and only if* $\{\coprod X_i \xrightarrow{\coprod f_i} X\}$ *is a Lichtenbaum covering.*

Proof. This is indeed a topology and is shown in [**17**] to be subcanonical. To see that $\{X_i \xrightarrow{f_i} U\}_{i \in I}$ is a Lichtenbaum covering if and only if $\coprod X_i \xrightarrow{\coprod f_i} U$ is a Lichtenbaum covering, note that if $x \in U$ has a neighborhood U_x and $s : U_x \to X_i$ is continuous with $f_i \circ s = \mathrm{id}$ then we can just extend $s : U_x \to X_i \to \coprod X_i$ and verify $(\coprod f_i) \circ s = \mathrm{id}$. If there is a local section $s : U_x \to \coprod X_i$ with $s(x) \in X_i$, then $s^{-1}(X_i)$ is a neighborhood of x on which $(\coprod f_i) \circ s = \mathrm{id}$, so we can restrict the target of s to X_i and the domain to $s^{-1}(X_i)$; then $f_i \circ s|s^{-1}(X_i) = \mathrm{id}$. □

Proposition 2.2.11. T_G^L *is saturated.*

Proof. Suppose $\{f_i : X_i \to X\}$ has a refinement by a Lichtenbaum covering $\{g_j : Y_j \to X\}$. Then for each j there is a map $h_j : Y_j \to X_{i(j)}$ such that $f_{i(j)} \circ h_j = g_j$. For each $x \in X$ there is a neighborhood U of x and a continuous map $s : U \to Y_j$ for some j such that $g_j \circ s = \mathrm{id}_U$. Then $h_j \circ s$ is a continuous map $U \to X_{i(j)}$ such that $f_{i(j)} \circ (h_j \circ s) = \mathrm{id}_U$, so $\{f_i : X_i \to X\}$ is a Lichtenbaum covering. □

2.2.1.5. *Other Topologies.*

DEFINITION 2.2.12. *Let T_G^{lh} be the topology whose coverings are epimorphic families $\{U_i \xrightarrow{f_i} U\}$ of local homeomorphisms, i.e. $U = \bigcup f_i(U_i)$ and for each i and each $x \in U_i$ there is a neighborhood V of x in U_i such that $f_i|V$ is a homeomorphism.*

It is easy to verify that a family $\{U_i \xrightarrow{f_i} U\}$ is a covering in T_G^{lh} if and only if $\{\coprod U_i \xrightarrow{\coprod f_i} U\}$ is, and that T_G^{lh} is a subcanonical topology. This topology resembles the étale topology.

Though we will not be working much with the following topology, we want to suggest it as another topology which may be used to produce a viable cohomology theory for G.

DEFINITION 2.2.13. *Let T_G^{oi} be the topology whose coverings are epimorphic families of open continuous injective maps, i.e. open embeddings.*

If $G = 1$, T_G^{oi} and T_G^{lh} are equivalent [**33**, Example 2.50], and each is equivalent to T_G [**8**, Lemma 1].

PROPOSITION 2.2.14. *Let A be an abelian group. For any of the topologies $T_G, T_G^s, T_G^{\mathrm{can}}$, T_G^o, T_G^L, T_G^{lh}, and T_G^{oi}, the constant sheaf A is the sheaf represented by A_d, where $A_d = A$ with discrete topology and trivial G-action.*

PROOF. Let P be the constant presheaf A on T_G. The map $P \to \mathrm{Hom}(-, A_d)$ that takes $a \in P(U) = A$ to the constant map with value a is an injection of presheaves, hence induces an injection of sheaves. To see that it is locally surjective, suppose $s \in \mathrm{Hom}(U, A_d)$. For each $a \in A$, let $U_a = s^{-1}(a) \subseteq U$. Since each U_a is a G-space which is an open subspace of U, the collection $\{U_a \hookrightarrow U\}$ is a covering in all of the topologies listed. Since s is constant on each U_a, this implies the map $P \to \mathrm{Hom}(-, A_d)$ is locally surjective, hence induces an epimorphism of the sheaves, in each of these topologies.

□

2.2.2. Comparison of the Topologies on C_G.

Proposition 2.2.15. *For any topological group G (even $G = 1$), we have $T_G^{oi} \prec T_G^{lh} \prec T_G^L \precneqq T_G^o \precneqq T_G^{\mathrm{can}} \precneqq T_G^s$.*
Therefore, $\mathcal{S}(T_G^{oi}) \supseteq \mathcal{S}(T_G^{lh}) \supseteq \mathcal{S}(T_G^L) \supsetneqq \mathcal{S}(T_G^o) \supsetneqq \mathcal{S}(T_G^{\mathrm{can}}) \supsetneqq \mathcal{S}(T_G^s)$. Also, $T_G^c \precneqq T_G^L$, so $\mathcal{S}(T_G^c) \supsetneqq \mathcal{S}(T_G^L)$.

Proof. An open injective map is a local homeomorphism, and each local homeomorphism has local sections, hence $T_G^{oi} \prec T_G^{lh} \prec T_G^L$, and we have already shown T_G^o is subcanonical, i.e. $T_G^o \prec T_G^{\mathrm{can}}$, and $T_G^{\mathrm{can}} \precneqq T_G^s$. Obviously, every covering in T_G^c is a covering in T_G^L. Now we show that every covering in T_G^L has a refinement in T_G^o. Let $\mathcal{U} = \{X_i \xrightarrow{f_i} X\}$ be a Lichtenbaum covering. For every $x \in X$, let U_x be an open set in X such that there is a (continuous) section $s_x : U_x \to X_{i(x)}$, so that $f_{i(x)} \circ s_x = \mathrm{id}_{U_x}$. We claim the family $\{G \times U_x \xrightarrow{a_x} X\}$ is an open cover which refines $\mathcal{U}$, where the map $a_x : G \times U_x \to X$ is given by $(g, u) \mapsto g \cdot u$. First note that each map is indeed open, since for any open subsets $V \subseteq U_x$, $W \subseteq G$, the image of $V \times W$ is $\bigcup_{g \in W} g \cdot V$, which is open. Of course, since $X = \bigcup U_x$, this family is epimorphic. This family is a refinement of $\mathcal{U}$ because for any x we define the map $U_x \times G \to X_{i(x)} : (u, g) \mapsto g \cdot s_x(u)$ and note $f_{i(x)}(g \cdot s_x(u)) = g \cdot f_{i(x)}(s_x(u)) = g \cdot u = a_x(u, g)$.

For our examples we will consider G-spaces with trivial G-action, so we may as well be working in the category of topological spaces with continuous maps. An example of an open covering which does not have a refinement by a local section covering is the map $q : X = \operatorname{Spec} \mathbb{Z} \to Y$ in the example in Section 2.1.3. Clearly, q is open, continuous, and surjective, but if it has a refinement by a Lichtenbaum covering then it is itself a Lichtenbaum covering (by Proposition 2.2.11), which is clearly not true.

An example of a canonical covering which does not have a refinement by an open covering is $\{U_1 \xrightarrow{f} U\}$, where $U_1 = \{a, b, b'\}$ with $\{b\}$ and $\{a, b\}$ the only nontrivial open sets, $U = \{a, b\}$ with trivial topology, and $f(a) = a, f(b) = f(b') = b$. Clearly, f is a quotient map; let us see why it is a universal quotient map. Let $V \xrightarrow{g} U$ be a continuous map. Then $V \times_U U_1 = g^{-1}(a) \times \{a\} \cup g^{-1}(b) \times \{b, b'\}$. Suppose $W \subseteq V$ is a subset such that the preimage under $\bar{f} : V \times_U U_1 \to V$ is open. We need to show W is open.

Note $\bar{f}^{-1}(W) = (g^{-1}(a) \cap W) \times \{a\} \cup (g^{-1}(b) \cap W) \times \{b, b'\}$. Now $g^{-1}(b) \cap W \times \{b, b'\}$ must be covered by sets of the form $(V' \times U') \cap V \times_U U_1$ where $V' \subseteq V, U' \subseteq U_1$ are open subsets. This means $\bar{f}^{-1}(W)$ is open if and only if (1) every point (w, a) with $w \in W \cap g^{-1}(a)$ is contained in an open set $(V_1 \times \{a, b\}) \cap (V \times_U U_1) \subseteq \bar{f}^{-1}(W)$ or an open set $(V_2 \times U_1) \cap (V \times_U U_1) \subseteq \bar{f}^{-1}(W)$ and (2) every point (w, b') with $w \in W \cap g^{-1}(b)$ is contained in a set $(V_3 \times U_1) \cap (V \times_U U_1)$, because under these conditions every point (w, b) with $w \in W \cap g^{-1}(b)$ is already covered by open sets. In turn (taking unions), this is equivalent to $\bar{f}^{-1}(W)$ being the union of $(V_1 \times \{a, b\}) \cap (V \times_U U_1)$, $(V_2 \times U_1) \cap (V \times_U U_1)$, and $(V_3 \times U_1) \cap (V \times_U U_1)$ for some open sets $V_1, V_2, V_3 \subseteq V$ such that (1) $(V_1 \times \{a, b\}) \cap (V \times_U U_1) = ((V_1 \cap g^{-1}(a)) \times \{a\}) \cup ((V_1 \cap g^{-1}(b)) \times \{b\}) \subseteq \bar{f}^{-1}(W)$, which means $V_1 \subseteq W$, (2) $(V_2 \times U_1) \cap (V \times_U U_1) = ((V_2 \cap g^{-1}(a)) \times \{a\}) \cup ((V_2 \cap g^{-1}(b)) \times \{b, b'\}) \subseteq \bar{f}^{-1}(W)$, which means $V_2 \subseteq W$, and (3) $(V_3 \times U_1) \cap (V \times_U U_1) = ((V_3 \cap g^{-1}(a)) \times \{a\}) \cup ((V_3 \cap g^{-1}(b)) \times \{b, b'\}) \subseteq \bar{f}^{-1}(W)$, which means $V_3 \subseteq W$. This means $W = V_1 \cup V_2 \cup V_3$ is a union of open sets, hence is open. Thus, f is a universal quotient map.

To show that $\{U_1 \xrightarrow{f} U\}$ does not have a refinement by an open cover, suppose $\{V_i \to U\}$ is such a refinement. Then $\{\coprod V_i \to U\}$ is also such a refinement and we may as well assume we have a refinement $\{V \xrightarrow{g} U\}$. By assumption, there is a continuous map $h : V \to U_1$ such that $f \circ h = g$. But then $h^{-1}(b)$ is open. If $h^{-1}(b) \neq \emptyset$, then $g(h^{-1}(b)) = \{b\}$ is open: a contradiction. This means $h^{-1}(b) = \emptyset$, so $h^{-1}(a) = h^{-1}(\{a, b\})$ is open and $g(h^{-1}(a)) = \{a\}$ is open: a contradiction (we cannot have $h^{-1}(a) = \emptyset$ since $h^{-1}(\{a\}) = g^{-1}(\{a\})$ and g is onto).

Finally, here is an example of a covering in T_G^{oi} (hence in T_G) that does not have a refinement in T_G^c: let $A = \{a\}, B = \{b\}, C = \{a, b\}$, where C is discrete. The inclusions

$A \hookrightarrow B, B \hookrightarrow C$ are open imbeddings, but there is no refinement of $\{A \hookrightarrow B, B \hookrightarrow C\}$ in T^c_G since we would have an epimorphism that factors through a map which is not an epimorphism, which is impossible. □

There are examples of coverings in T^c_G which are not open coverings but can be refined by open coverings: consider, for example $\{X \xrightarrow{f} Y\}$, where $X = \{a, a', b\}$ with $\{a'\}$ the only nontrivial open set, $Y = \{a, b\}$ with the trivial topology, and $f(a) = f(a') = a, f(b) = b$. Clearly, the map $s : a \mapsto a, b \mapsto b$ is a section, but the image of the open set $\{a'\}$ is not open.

Proposition 2.2.16. *The topology T^{lh}_G is equivalent to T^L_G if and only if G has the discrete topology.*

Proof. If G has any topology other than the discrete one, then the covering $\mathcal{U} = \{G \to pt\}$ is in T^L_G, but it does not have a refinement that is in T^{lh}_G: if $X \to pt$ is any map in a refinement of $\mathcal{U}$, as a G-set X must be a disjoint union of orbits of the form G, and by Lemma 1.3.2 each orbit must have a subspace topology that is no finer than than the original topology of G, which is not discrete. This means $X \to pt$ cannot be a local homeomorphism, because no point of X can be open.

If G has the discrete topology and $\mathcal{U} = \{X_i \xrightarrow{f_i} X\}$ is a Lichtenbaum covering then for every $x \in X$ there is a neighborhood U_x and a continuous section $s_x : U_x \to X_{i(x)}$. The family $\{G \times U_x \xrightarrow{a} X\}_{x \in X}$, where $a(g, u) = g \cdot u$, is a refinement of $\mathcal{U}$ and a covering in T^{lh}_G. Indeed, a is a local homeomorphism because the point (g, u) has the neighborhood $\{g\} \times U_x$, which maps homeomorphically onto $g \cdot U_x$ by a. □

Proposition 2.2.17. *If $G \neq 1$ has the discrete topology, then $T^{oi}_G \lneqq T^{lh}_G$. In general, if G has a subgroup $H \neq G$ such that (1) $G = H \times H'$ for some subgroup H' of G and (2) H contains no proper open subgroup, then $T^{oi}_G \lneqq T^{lh}_G$.*

Proof. The covering $\{G \xrightarrow{\pi_1} H\}$, where $\pi_1(h, h') = h$ for $h \in H, h' \in H'$, does not have a refinement by a family of (continuous) open injective maps: any such map would have to map onto all of H, hence be an isomorphism, and there is no section

$H \to G$ that is G-equivariant unless $H' = 1$ (in which case $H = G$, contradicting the hypothesis of the proposition). But $\pi_1 : G \to H$ is a local homeomorphism: for any $g = (h, h') \in G$, the set $gH = \{(h, h') \mid h \in H\}$ is open and maps homeomorphically onto H. If G is discrete, we can take $H = \{e\}$. □

One nontrivial example where the above proposition applies is when $G = \mathbb{Z}/6\mathbb{Z}$, with the topology where (2) and $1 + (2)$ are the only nontrivial open sets and $H = (2)$.

Corollary 2.2.18. *For any group $G \neq 1$, $T_G^{oi} \lneq T_G^L$.*

2.2.3. Čech Cohomology. The following result is easy, but is important enough to be called a theorem. It is only a slight generalization of [**17**, Prop. 1.4]:

Theorem 2.2.19. *Suppose $\{G \to pt\}$ is a covering in a subcanonical topology T on $\mathbf{C}_G$ (which is the case for all the subcanonical topologies considered above) and A is any topological G-module. Let $\tilde{A}$ be the sheaf represented by A. The Čech cohomology $\check{H}^n(T, pt, \tilde{A})$ is canonically isomorphic to $H_c^n(G, A)$.*

Proof. The Čech cohomology cochains $\check{C}^n(pt, \tilde{A})$ are defined as the direct limit of $\prod \tilde{A}(U_{i_0} \times_{pt} \cdots \times_{pt} U_{i_n})$ over all coverings $\{U_i \to pt\}$ of pt. But the covering $\{G \to pt\}$ is cofinal in all coverings because for any G-space X and any $x \in X$ the map $G \to X : g \mapsto g \cdot x$ is continuous. That means $\check{C}^n(pt, \tilde{A}) = \mathrm{Hom}(G^{n+1}, A)$ is the set of homogeneous cochains in the continuous cochain theory. Since the coboundary morphisms are the same, the two cohomology theories coincide by Theorem 2.1.1. □

Corollary 2.2.20. *Suppose $\{G \to pt\}$ is a covering in a subcanonical topology T on $\mathbf{C}_G$. Then $H^1(T, pt, \tilde{A}) \cong H_c^1(G, A)$ for any topological G-module A.*

Proof. $H^1(T, pt, \tilde{A}) \cong \check{H}^1(T, pt, \tilde{A})$. □

2.2.4. Coverings with Single Maps.

Definition 2.2.21. *Let T be a topology on a category $\mathbf{C}$ which has arbitrary coproducts. We define T^1 to be the topology whose coverings are single-morphism coverings in T.*

It is easy to check T^1 is a topology if T is. Throughout this section, we will use the following properties of a Grothendieck topology on $\mathcal{C}$:

(C) T is saturated and for any set $\{X_i\}_{i\in I}$ of objects in $\mathcal{C}$, the family of natural inclusions $\{X_i \hookrightarrow \coprod X_i\}_{i\in I}$ is a covering.

(C′) $\{f_i : X_i \to X\}$ is a covering in T if and only if $\{\coprod f_i : \coprod X_i \to X\}$ is a covering in T.

For example, T_G^L and T_G^{can} satisfy (C); and $T_G^L, T_G^o, T_G^{\mathrm{lh}}, T_G^{\mathrm{can}}$ all satisfy (C′). Note that if T satisfies (C′) then for any family $\{X_i\}$ of objects of $\mathcal{C}$, $\{\mathrm{id} : \coprod X_i \to \coprod X_i\}$ is a covering by definition of a topology, which means $\{X_i \hookrightarrow \coprod X_i\}$ is a covering in T. Therefore, if T satisfies (C′) then its saturation satisfies (C).

There is an obvious morphism of topologies $\alpha : T^1 \to T$ which is the identity on $\mathcal{C}$. First of all, if T satisfies (C) or (C′) then the Čech cohomologies are the same:

Lemma 2.2.22. *Suppose T satisfies (C) or (C′). Then for all $n \geq 0$, all sheaves F on T, and all X in $\mathcal{C}$, we have $\check{H}^n(T, X, F) \cong \check{H}^n(T^1, X, \alpha_* F)$.*

Proof. Note F and α_*F are the same presheaf, and by definition any T^1 covering is a T covering so there is a natural map ϕ_n from $\check{H}^n(T^1, X, F)$ to $\check{H}^n(T, X, F)$. Because F is a sheaf on T, which satisfies (C) or (C′), for any family $\{X_i\}$ of objects in $\mathcal{C}$, $\{X_i \hookrightarrow \coprod X_i\}$ is a covering, so $\prod F(X_i) \cong F(\coprod X_i)$. This means for any covering $\{X_i \to X\}_{i\in I}$ in T, the corresponding Čech complex $(\check{C}^n(\{X_i \to X\}, F))_{n=0}^{\infty}$ is isomorphic to the Čech complex $(\check{C}^n(\{\coprod X_i \to X\}, F))_{n=0}^{\infty}$. Now $\{\coprod X_i \to X\}$ is a covering in T^1: this is obvious if T satisfies (C′), and if (T) satisfies (C) then this is because the covering $\{\coprod X_i \to X\}$ has a refinement in T, so since T is saturated, $\{\coprod X_i \to X\}$ is in T, hence also in T^1. Therefore, the contribution to the direct limit

$$\check{H}^n(T, X, F) = \varinjlim_{\{X_i \to X\}} \check{H}^n(\{X_i \to X\}, F)$$

from the covering $\{X_i \to X\}$ also exists in the direct limit

$$\check{H}^n(T^1, X, F) = \varinjlim_{\{Y \to X\}} \check{H}^n(\{Y \to X\}, F)$$

This implies ϕ_n is an isomorphism. $\square$

Theorem 2.2.23. *Suppose T satisfies (C) or (C′). Then α_* is exact, so for any sheaf F on T and any X in C we have $H^n(T^1, X, \alpha_* F) \cong H^n(T, X, F)$.*

Proof. There are two proofs. *Proof 1.* We show $R^1\alpha_* F = 0$ for all sheaves F on T. $R^1\alpha_* F$ is the sheaf associated to the presheaf $P : X \mapsto H^1(T, \alpha X, F) = H^1(T, X, F)$ on T^1. But $H^1(T, X, F) \cong \check{H}^1(T, X, F) \cong \check{H}^1(T^1, X, F) \cong H^1(T^1, X, F)$, so this is precisely the sheaf $R^1(\mathrm{id}_{T^1})_* F = 0$.

Proof 2. We show α_* is exact directly. Suppose we have an epimorphism of sheaves $\eta : F_1 \to F_2$ on T. We want to show $\alpha_*\eta : \alpha_* F_1 \to \alpha_* F_2$ is locally surjective. So take a section $s \in \alpha_* F_2(X) = F_2(X)$. Since η is locally surjective, there exist a covering $\{f_i : X_i \to X\}$ in T and sections $s_i \in F_1(X_i)$ such that $s|X_i = \eta(X_i)(s_i)$. But since T satisfies (C) or (C′), $\prod F_2(X_i) \cong F_2(\coprod X_i)$ and $(s|X_i) \in \prod F_2(X_i)$ corresponds to $s|\coprod X_i$. Also, $(s_i) \in \prod F_1(X_i)$ corresponds to some element $t \in F_1(\coprod X_i)$. Because T satisfies (C) or (C′), $\{\coprod f_i : \coprod X_i \to X\}$ must be a covering in T^1, and $\eta(\coprod X_i)(t) = s|\coprod X_i$. Thus, $\alpha_*\eta$ is an epimorphism. $\square$

Example. Recall that T_G is the usual topology on the category of G-sets whose coverings are surjective families $\{X_i \to X\}$. Let T_G^1 be the restriction of T_G to coverings which consist of single morphisms. Then, by the proof of Proposition 2.2.14, the constant sheaf $\mathbb{Z}$ on T_G is the sheaf $\mathrm{Hom}(-, \mathbb{Z})$ represented by $\mathbb{Z}$ with trivial G-action, but the constant sheaf $\mathbb{Z}$ on T_G^1 is the constant presheaf $\mathbb{Z}$ by Proposition 2.1.22. The cohomology of the first is

$$H^n(T_G^1, pt, \mathrm{Hom}(-, \mathbb{Z})) \cong H^n(T_G, pt, \mathrm{Hom}(-, \mathbb{Z})) \cong H^n(G, \mathbb{Z}),$$

the usual group cohomology, by Theorem 2.2.23. The cohomology of the second is given by $H^0(T_G^1, pt, \mathbb{Z}) = \mathbb{Z}$ and $H^n(T_G^1, pt, \mathbb{Z}) = 0$ for $n > 1$ by the proof of Proposition 2.1.23.

2.2.5. The Use of Pointed G-Spaces.

Definition 2.2.24. *Let $C_{G,*}$ be the category of pointed G-spaces (X, x) (where X is a G-space and $x \in X$) with morphisms $f : (X, x) \to (Y, y)$ that are continuous G-maps satisfying $f(x) = y$.*

Definition 2.2.25. *Let $T_G^\bullet$ be a topology on C_G. We define $T_{G,*}^\bullet$ to be the topology on $C_{G,*}$ whose coverings are single maps $\{f : (X, x) \to (Y, y)\}$ which are coverings in $T_G^\bullet$.*

Note that there is a forgetful functor $g : C_{G,*} \to C_G$ which induces a morphism of topologies $T_{G,*}^\bullet \to T_G^{\bullet,1}$, where $T_G^{\bullet,1}$ is the restriction of $T_G^\bullet$ to coverings which consist of single morphisms, since g takes coverings to coverings. Consider the following property of a topology $T_G^\bullet$ on C_G:

(S) For any covering $\{f : X \to Y\}$, f is surjective as a map of sets.

Theorem 2.2.26. *Let $T_G^\bullet$ be a topology on C_G satisfying (S). Then for any sheaf F on $T_{G,1}^*$, we have $H^n(T_{G,*}^\bullet, pt, g_*F) \cong H^n(T_G^{\bullet,1}, pt, F)$.*

Proof. Since a covering in $T_G^{\bullet,1}$ is a surjective map, for any pointed G-space (X, x) and any covering $\{f : Y \to X\}$ in $T_G^{\bullet,1}$, there exists $y \in Y$ with $f(y) = x$, so $\{f : (Y, y) \to (X, x)\}$ is a covering in $T_{G,*}^\bullet$ which maps onto the covering $\{f : Y \to X\}$. Therefore, the theorem follows by Lemma 1.1.6. ☐

2.2.6. Comparison of $H^n_{ss}(G, A)$ to the Cochain Theories. In this section we show how Grothendieck topologies enable us to compare the semisimplicial theory $H^n_{ss}(G, A)$ of Wigner, in the form $H^n(T_G^L, pt, \tilde{A})$, to Moore's measurable cochain theory $H^n_m(G, A)$, in the form $H^n(T_G^m, pt, \tilde{A})$. The comparison will hold under only the assumption that G is second countable.

Lemma 2.2.27. *Suppose $\{f : X \to Y\}$ is a Lichtenbaum covering. If Y is Lindelöf*[3]*, then there is a global measurable section $s : Y \to X$.*

[3] A topological space is Lindelöf if any open covering has a countable subcovering.

Proof. The covering $\{U_y\}_{y\in Y}$ has a countable subcovering since Y is Lindelöf, say $\{U_{y_k}\}_{k=1}^{\infty}$. Let s_k be the corresponding section $s_k : U_{y_k} \to X$, let $U_1 = U_{y_1}$, and let $U_i = U_{y_i} \setminus \bigcup_{k=1}^{i-1} U_{y_k}$. Clearly, $\{U_k\}_{k=1}^{\infty}$ is a disjoint covering for Y and each U_k is measurable. Define s by $s(y) = s_k(y)$ if $y \in U_k$. Then for any measurable set V in X, $s^{-1}(V) = \bigcup_{k=1}^{\infty}(s_k^{-1}(V) \cap U_k)$ is the countable union of measurable sets, hence is measurable. □

Let $T_{G,sc}^{L,1}$ be the topology on the category of second-countable G-spaces whose coverings are Lichtenbaum coverings consisting of single maps $\{f : X \to Y\}$. Note that the final object pt and fibered products exist in the category of second-countable spaces. This is because subspaces of second countable spaces are second countable, and finite (even countable) products of second countable spaces are second countable. We have an obvious map of topologies $i : T_{G,sc}^{L,1} \to T_G^{L,1}$ which is just the embedding of categories.

Proposition 2.2.28. *Suppose G is second countable. For any sheaf F on $T_G^{L,1}$ and any second countable G-space X, we have canonical isomorphisms $H^n(T_{G,sc}^{L,1}, X, i_*F) \cong H^n(T_G^{L,1}, X, F)$ for all n.*

Proof. By Lemma 1.1.6, it is enough to show that any covering $\{f : X \to Y\}$ of a second countable G-space Y by an arbitrary G-space X is refinable by a covering $\{U \to Y\}$ with U second countable. For every point $y \in Y$ there is a neighborhood U_y of Y and a section $s_y : U_y \to X$. Note that U_y is second countable, and so is $G \times U_y$ (where U_y has trivial G-action). We have the two continuous G-maps $a_y : G\times U_y \to Y : (g, x) \mapsto g \cdot x$ and $t_y : G\times U_y \to X : (g, x) \mapsto g \cdot s_y(x)$, and $f \circ t_y = a_y$. Since any second countable space is Lindelöf, there is a countable subset $\{y_i\}_{i=1}^{\infty}$ of y's such that $Y = \bigcup_{i=1}^{\infty} U_{y_i}$. This implies $U = \coprod_{i=1}^{\infty} U_{y_i}$ is a second countable space. The induced maps $\coprod a_{y_i} : G \times U \to Y$, $\coprod t_{y_i} : G \times U \to X$ are still continuous and $f \circ (\coprod t_{y_i}) = \coprod a_{y_i}$. Furthermore, $\{G \times U \xrightarrow{\coprod a_{y_i}} Y\}$ is a covering in $T_G^{L,1}$ since for any $y \in Y$ there exists i with $y \in U_{y_i}$ and a section $s : U_{y_i} \to G \times U : x \mapsto (e, x)$. Thus, $\{G \times U \xrightarrow{\coprod a_{y_i}} Y\}$ is a refinement of $\{f : X \to Y\}$. □

Corollary 2.2.29. *Let G be a second countable group and A any topological G-module. Then* $\mathrm{Hom}_{C_G}(-, A)$ *is a sheaf on* $T^{L,1}_{G,sc}$ *and*

$$H^n(T^{L,1}_{G,sc}, pt, \mathrm{Hom}_{C_G}(-, A)) \cong H^n(T^{L,1}_G, pt, \tilde{A})$$

Note that, strictly speaking, $\mathrm{Hom}_{C_G}(-, A)$ is not in general the sheaf represented by A on $T^{L,1}_{G,sc}$ because A is not necessarily second countable, hence not an object in $\mathrm{Cat}(T^{L,1}_{G,sc})$, but $\mathrm{Hom}_{C_G}(-, A) = i_*\tilde{A}$ is still a sheaf on $T^{L,1}_{G,sc}$.

Proposition 2.2.30. *Let G be second countable. Then there is a morphism of topologies* $j : T^{L,1}_{G,sc} \to T^m_G$ *such that for any sheaf F on* T^m_G *and any G-space X,* $H^n(T^{L,1}_{G,sc}, X, j_*F) \cong H^n(T^m_G, X, F)$ *for all n.*

Proof. The morphism j takes a G-space to itself (note continuous maps are measurable). Note j is a morphism of topologies because fibered products are the same in both categories and a covering $\{f : X \to Y\}$ in $T^{L,1}_{G,sc}$ is a covering in T^m_G by Lemma 2.2.27. As shown in Remark 1 in Section 2.1.2, any covering $\{f : X \to Y\}$ in T^m_G has a refinement by the covering $\{G \times Y_{\mathrm{triv}} \to Y\}$, which has a global continuous section hence is a covering in $T^{L,1}_{G,sc}$. Thus the proposition holds by Lemma 1.1.6. □

Corollary 2.2.31. *Let G be a second countable group and A any topological G-module. Then* $\mathrm{Hom}_{C^m_G}(-, A)$ *is a sheaf on* $T^{L,1}_{G,sc}$ *and* $H^n(T^{L,1}_{G,sc}, pt, \mathrm{Hom}_{C^m_G}(-, A)) \cong H^n(T^m_G, pt, \tilde{A}) \cong H^n_m(G, A)$ *for all n.*

Theorem 2.2.32. *Let G be a second countable group and A any topological G-module. There exist canonical maps* $\phi_n : H^n_{ss}(G, A) \to H^n_m(G, A)$ *for all n.*

Proof. First we use Lichtenbaum's Theorem that $H^n_{ss}(G, A) \cong H^n(T^L_G, pt, \tilde{A})$. Next, by Proposition 2.2.28 and Theorem 2.2.23, we have

$$H^n(T^L_G, pt, \tilde{A}) \cong H^n(T^{L,1}_{G,sc}, pt, \mathrm{Hom}_{C_G}(-, A))$$

(we use the notation $\operatorname{Hom}_{C_G}(-,A)$ because technically, A may not be an object in the category underlying $T^{L,1}_{G,sc}$). Now Proposition 2.2.30 implies $\operatorname{Hom}_{C^m_G}(-,A)$ is a sheaf on $T^{L,1}_{G,sc}$ and $H^n(T^{L,1}_{G,sc}, pt, \operatorname{Hom}_{C^m_G}(-,A)) \cong H^n(T^m_G, pt, \tilde{A}) \cong H^n_m(G,A)$. Now the natural injection of sheaves $\operatorname{Hom}_{C_G}(-,A) \hookrightarrow \operatorname{Hom}_{C^m_G}(-,A)$ on $T^{L,1}_{G,sc}$ induces the maps ϕ_n on cohomology. □

There is even a long exact sequence connecting the cohomology groups $H^n_{ss}(G,A)$ and $H^n_m(G,A)$:

$$\cdots \to H^n_{ss}(G,A) \to H^n_m(G,A) \to H^n_Q(G,A) \to H^{n+1}_{ss}(G,A) \to \cdots$$

where $H^n_Q(G,A) = H^n(T^{L,1}_{G,sc}, pt, Q)$ and Q is the sheaf $\operatorname{Hom}_{C^m_G}(-,A)/\operatorname{Hom}_{C_G}(-,A)$.

The same reasoning could be applied to any category of Lindelöf spaces, not just the second countable spaces, as long as the category has fibered products.

Let $T^L_{G,sc,*}$ be the topology on the category of pointed second countable G-spaces whose coverings are those in $T^L_{G,*}$, i.e. Lichtenbaum coverings consisting of single maps $\{f : (X,x) \to (Y,y)\}$ such that there is a section s of f on a neigborhood U of y with $s(y) = x$. Note that the final object pt and fibered products exist in the underlying category. There is an obvious morphism $t : T^L_{G,sc,*} \to T^{L,1}_{G,sc}$ forgetting the base point x in (X,x). By the proof of Theorem 2.2.26, we have

Proposition 2.2.33. *$H^n(T^L_{G,sc,*}, (X,x), t_*F) = H^n(T^{L,1}_{G,sc}, X, F)$ for any sheaf F on $T^{L,1}_{G,sc}$ and any second countable pointed G-space (X,x).*

Corollary 2.2.34. *Let G be a second countable group and A a topological G-module. Then $\operatorname{Hom}_{C_G}(-,A)$ is a sheaf on $T^L_{G,sc,*}$ and*

$$H^n(T^L_{G,sc,*}, pt, \operatorname{Hom}_{C_G}(-,A)) \cong H^n(T^{L,1}_{G,sc}, pt, \tilde{A}) \cong H^n(T^{L,1}_G, pt, \tilde{A}).$$

Proposition 2.2.35. *Let G be second countable. Then there is a morphism of topologies $j : T^L_{G,sc,*} \to T^{\mathrm{lcm}}_G$ such that for any sheaf F on T^{lcm}_G and any pointed G-space (X,x), $H^n(T^L_{G,sc,*}, (X,x), j_*F) \cong H^n(T^{\mathrm{lcm}}_G, (X,x), F)$ for all n.*

Proof. The proof is the same as that of Proposition 2.2.30. □

Recall $\mathrm{Map}_{G,\mathrm{lcm}}((X,x),A)$ is the set of measurable G-maps from X to A that are continuous in a neighborhood of x.

Corollary 2.2.36. *Let G be a second countable group and A a topological G-module. Then $\mathrm{Map}_{G,\mathrm{lcm}}(-,A)$ is a sheaf on $T^L_{G,sc,*}$ and*

$$H^n(T^L_{G,sc,*}, pt, \mathrm{Map}_{G,\mathrm{lcm}}(-,A)) \cong H^n(T^{\mathrm{lcm}}_G, pt, \mathrm{Map}_{G,\mathrm{lcm}}(-,A)) \cong H^n_{\mathrm{lcm}}(G,A)$$

for all n.

Theorem 2.2.37. *Let G be a second countable group A a topological G-module. There exist canonical maps $\phi_n : H^n_{ss}(G,A) \to H^n_{\mathrm{lcm}}(G,A)$ for all n.*

Proof. First we use Lichtenbaum's Theorem that $H^n_{ss}(G,A) \cong H^n(T^L_G, pt, \tilde{A})$. Next, by Proposition 2.2.28 and Theorem 2.2.23, we have

$$H^n(T^L_G, pt, \tilde{A}) \cong H^n(T^{L,1}_{G,sc}, pt, \mathrm{Hom}_{C_G}(-,A)).$$

Corollary 2.2.34 implies

$$H^n(T^{L,1}_{G,sc}, pt, \mathrm{Hom}_{C_G}(-,A)) \cong H^n(T^L_{G,sc,*}, pt, \mathrm{Hom}_{C_G}(g(-),A)),$$

where $g(X,x) = X$. In total, so far we have

$$H^n_{ss}(G,A) \cong H^n(T^L_{G,sc,*}, pt, \mathrm{Hom}_{C_G}(g(-),A)).$$

Now Corollary 2.2.36 implies $\mathrm{Map}_{G,\mathrm{lcm}}(-,A)$ is a sheaf on $T^L_{G,sc,*}$ and

$$H^n(T^L_{G,sc,*}, pt, \mathrm{Map}_{G,\mathrm{lcm}}(-,A)) \cong H^n(T^{\mathrm{lcm}}_G, pt, \mathrm{Map}_{G,\mathrm{lcm}}(-,A)) \cong H^n_{\mathrm{lcm}}(G,A).$$

Now the natural injection of sheaves $\mathrm{Hom}_{C_G}(g(-),A) \hookrightarrow \mathrm{Map}_{G,\mathrm{lcm}}(-,A)$ on $T^L_{G,sc,*}$ induces the maps ϕ_n on cohomology. □

Theorem 2.2.38. *Let G be a second countable group A a topological G-module. There exist canonical maps $\phi_n : H^n_{ss}(G,A) \to H^n_{\mathrm{lc}}(G,A)$ for all n.*

Proof. The proof is the same - just change "lcm" into "lc" in Proposition 2.2.35, Corollary 2.2.36, and Theorem 2.2.37. □

2.3. Associated Short Exact Sequences

In this section we first discuss different Yoneda embeddings of the category $\mathcal{M}_G$ of topological G-modules in various categories of sheaves; abusing notation, we denote them all by y. Then we explain how these embeddings enable us to connect these categories of sheaves to short exact sequence of topological G-modules which give long exact sequences in the associated cohomology theories.

If T is a subcanonical topology on C_G, there is a Yoneda embedding $y : \mathcal{M}_G \to \mathcal{S}(T) : A \mapsto \tilde{A}$, which is fully faithful. For a Grothendieck topology T on $C_{G,*}$, such an embedding $y : \mathcal{M}_G \to \mathcal{S}(T)$ also exists; here $y(A)$ is represented by $(A, 0)$. The Yoneda embedding is still fully faithful in this case because any morphism of two representable sheaves comes from a unique morphism of topological G-modules $A \to B$, and such morphisms do take 0 to 0. In both cases, the Yoneda embedding is left exact (and in particular, exact in the middle).

Definition 2.3.1. *For either of the above situations, let T be the respective topology and $\mathcal{S} = \mathcal{S}(T)$ be the category of sheaves. Define a class S_T of morphisms in $\mathcal{M}_G$ by $S_T = y^\#(\mathrm{Mor}(\mathcal{S}))$, where $y^\#$ is the Yoneda pullback discussed in Section 1.2.3.*

By definition, S_T consists of the proper morphisms $\phi : A \to B$ of topological G-modules such that the exact sequences

$$0 \to \ker\phi \to A \to \operatorname{Im}\phi \to 0$$

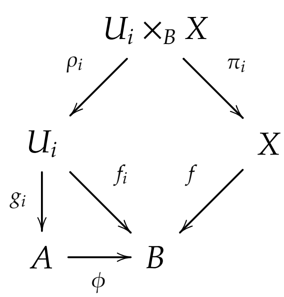
and

$$0 \to \operatorname{Im}\phi \to B \to \operatorname{coker}\phi \to 0$$

are sent to exact sequences in $\mathcal{S}$. Since y is left-exact, this amounts to saying $\phi \in S_T \iff \phi$ is proper, $y(A) \to y(\operatorname{Im}\phi)$ is an epimorphism and $y(B) \to y(\operatorname{coker}\phi)$ is an epimorphism. Note that by Theorem 1.2.9, $(\mathcal{M}_G, S_T)$ *is a quasi-abelian S-category*. The following proposition, together with Proposition 1.2.1 of Section 1.2.1, gives an explicit description of S_T in terms of the coverings in T.

Proposition 2.3.2. *The class of epimorphisms in S_T is precisely the class of proper epimorphisms $\phi : A \to B$ such that there is a covering $\{U_i \xrightarrow{f_i} B\}_{i \in I}$ in T refining $\{A \xrightarrow{\phi} B\}$ (where the U_i, A, and B may be pointed spaces, depending on the category underlying T).*

Proof. If ϕ is a proper epimorphism then $\operatorname{coker}\phi = 0$ so $y(B) \to y(\operatorname{coker}\phi)$ is automatically an epimorphism; we have to show that if $\{\phi : A \to B\}$ has a refinement, then $y(A) \to y(\operatorname{Im}\phi) = y(B)$, which is the map $y(\phi)$, is an epimorphism in $\mathcal{S}$. By Lemma 1.1.5, it is enough to show $y(\phi)$ is locally surjective. First suppose we are working with a topology T on C_G. If there is a refinement $\{U_i \xrightarrow{f_i} B\}$ of $\{\phi : A \to B\}$, then for each $i \in I$ there is a map $g_i : U_i \to A$ such that $\phi \circ g_i = f_i$ (see the diagram below). For any map $f : X \to B$, $f \circ \pi_i = \phi \circ g_i \circ \rho_i$, which means the restriction of the element $f \in \tilde{B}(X)$ to $U_i \times_B X$ is the image of $g_i \circ \rho_i \in \tilde{A}(U_i \times_B X)$ under $y(\phi)$. Since $\{U_i \times_B X \xrightarrow{\pi_i} X\}$ is a covering in T (being the pullback of the covering $\{U_i \xrightarrow{f_i} B\}$ by $f : X \to B$), this implies $y(\phi)$ is locally surjective.



If, instead, we are working with a topology T on $C_{G,*}$ and we have a refinement $\{(U_i, u_i) \xrightarrow{f_i} B\}$ of $\{\phi : A \to B\}$ then for any map $f : (X, x) \to (B, 0)$ we can form the fibered products $(U_i \times_B X, (u_i, x))$ and the same argument works.

Conversely, if $\phi : A \to B$ is an epimorphism in S_T then, by definition of S_T, ϕ is proper and $y(A \to \operatorname{Im}(\phi)) = y(\phi)$ is an epimorphism, hence is locally surjective, so id_B is locally in the image of $y(A)$. This means there exist a covering $\{U_i \xrightarrow{f_i} B\}_{i \in I}$ and maps $g_i : U_i \to A$ such that $\phi \circ g_i = f_i$, so $\{U_i \to B\}$ is a refinement of $\{A \xrightarrow{\phi} B\}$ (the U_i, A, and B are pointed if we are working with a topology on $C_{G,*}$). □

Now we can prove that the Yoneda functors preserve and reflect exactness in S_T:

Proposition 2.3.3. *Suppose $\beta : B \to C$ is a proper epimorphism. Then the following are equivalent:*

(1) The sequence $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ in $\mathcal{M}_G$ is exact and $\alpha, \beta \in S_T$.

(2) The sequence $0 \to y(A) \xrightarrow{y(\alpha)} y(B) \xrightarrow{y(\beta)} y(C) \to 0$ is exact in $\mathcal{S}$.

Proof. (1) $\Rightarrow$ (2): The Yoneda embedding y is left-exact, and if β is in S_T, then $y(B \to \mathrm{Im}(\beta)) = y(\beta)$ is an epimorphism. (2) $\Rightarrow$ (1): For any map $f : X \to B$ (where X is a topological G-module) with $\beta \circ f = 0$, there is a unique map $g : X \to A$ such that $\alpha \circ g = f$, so $\alpha = \ker(\beta)$, hence the sequence in $\mathcal{M}_G$ is exact. Now $\beta \in S_T$ since $y(B \to \mathrm{Im}(\beta)) = y(\beta)$ is an epimorphism. Finally, $\alpha = \ker\beta$ is also in S_T since $(\mathcal{M}_G, S_T)$ is an S-category. □

Corollary 2.3.4. *For any exact sequence $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ in $(\mathcal{M}_G, S_T)$ there is a long exact sequence on cohomology*

$$0 \to H^0(T, pt, \tilde{A}) \to H^0(T, pt, \tilde{B}) \to H^0(T, pt, \tilde{C}) \to H^1(T, pt, \tilde{A}) \to \cdots$$

Proof. By Proposition 2.3.3 we have a short exact sequence of sheaves, hence by the theory of derived functors, a long exact sequence in cohomology. □

In the following table we describe the classes S_T for various topologies T on C_G and $C_{G,*}$ by describing the corresponding classes of proper epimorphisms $e : A \to B$ in S_T.

Topology T	**Epimorphisms** $e : A \to B$ **in** S_T
T_G^c	e has a global continuous section
T_G^L and $T_G^{L,1}$	e has local sections
T_G^o and T_G^{can}	all e

For the proof, by Proposition 2.3.2 we just have to show that the epimorphisms which have a refinement in T are precisely the ones listed. For T_G^c, any epimorphism which has a continuous section is itself a covering. Conversely, if $\{e : A \to B\}$ has a refinement by a cover $\{X \xrightarrow{f} B\}$, then f has a continuous section $s : B \to X$

with $f \circ s = \mathrm{id}_B$. Since f factors through A, say $f = e \circ g$ for some continuous map $g : X \to A$, $g \circ s$ is a continuous section of e. The proof for T_G^L and $T_G^{L,1}$ is also the same, noting that if $\{e : A \to B\}$ has a refinement by a cover $\{U_i \to B\}$ then it has a refinement by the cover $\{\coprod U_i \to B\}$. For T_G^o and T_G^{can}, note that any proper epimorphism is an open surjective continuous map, hence a covering in both topologies.

CHAPTER 3

$\mathrm{Ext}^n_{(\mathcal{M}_G,S)}(\mathbb{Z}, A)$ and Cochain Theories

This chapter is devoted to discussing the two most basic cochain theories, the continuous cochain theory and the measurable cochain theory, and verifying some of Wigner's results in [**35**].

3.1. Continuous Cochain Cohomology

This section is a reworking of David Wigner's section on the continuous cochain theory[**35**, p. 86], which was defined in Section 2.1.1.1.

In this section, let S be the class of morphisms such that the epimorphisms in S are proper and have a continuous section (see Section 1.2.3). A short exact sequence in S is of the form $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ such that there is a continuous map $\gamma : C \to B$ with $\beta \circ \gamma = \mathrm{id}_C$. We claim that such a short exact sequence gives a long exact sequence on cohomology

$$0 \to H^0_c(G, A) \to H^0_c(G, B) \to H^0_c(G, C) \to H^1_c(G, A) \to \cdots$$

First, the sequence $0 \to C^n(G, A) \to C^n(G, B) \to C^n(G, C) \to 0$ is exact: the map $C^n(G, A) \to C^n(G, B)$ is obviously injective, and for a map $f \in C^n(G, C)$, the map $\gamma \circ f \in C^n(G, B)$ maps to f, so the map $C^n(G, B) \to C^n(G, C)$ is surjective. If we have a map $f : G^n \to B$ such that $\beta \circ f = 0$ then there exists a unique map $h : G^n \to A$ such that $\alpha \circ h = f$. To show h is continuous, suppose we have an open set $U \subseteq A$. Then, since α is proper, $\alpha(U) = U' \cap \alpha(A)$ for some open set $U' \subseteq B$, and $h^{-1}(U) = f^{-1}(U')$ is open. Now the sequence on cohomology is exact by the Snake Lemma.

Let $F(G, A) = C^1(G, A)$, made into a G-module by $(g \cdot f)(x) = f(xg)$, $(f_1 + f_2)(x) = f_1(x) + f_2(x)$, and given the compact-open topology, i.e. a sub-basis for the topology on $F(G, A)$ is the set of all subsets $N(K, U) = \{f \in F(G, A) \mid f(K) \subseteq U\}$ where K is a compact subset of G and U is an open subset of A.

To show that $F(G,A)$ is a topological G-module, we need to assume that G is locally compact, which is what we will do throughout the rest of this section. Since a topological group is regular (see Section 1.3.1), this implies that for every neighborhood U of $x \in G$ there is a compact set K containing a neighborhood of x with $K \subseteq U$ (by local compactness, there is a compact set K_1 containing a neighborhood U_1 of x; by regularity, there is an open set V with $x \in V \subseteq \bar{V} \subseteq U_1 \cap U$; $\bar{V}$ is a closed subspace of the compact space K_1, so $K := \bar{V}$ is compact).

To show that the action of G is continuous, suppose $g \cdot f \in N(K,U)$, i.e. $f(Kg) \subseteq U$. For every $x \in K$ there is an open set $P_x \ni x$ and an open set $Q_x \ni g$ with $P_x \cdot Q_x \subseteq f^{-1}(U)$. These P_x cover K, so we can choose finitely many of them, say $P_1, \dots, P_n$, that cover K. The intersection Q of the corresponding Q_x's contains an open set $V \ni g$ which is inside an open set K', which is itself inside Q. Then KK' is compact (being the image of $K \times K'$ under $G \times G \to G$), $N(KK',U)$ is a neighborhood of f (for any $k \in K, k' \in K' \subseteq Q$ we have $f(kk') \in U$ since $k \in P_i$ for some i and $f(P_i \cdot Q_i) \subseteq U$), and V is a neighborhood of g such that $V \cdot N(KK',U) \subseteq N(K,U)$ (for any $f \in N(KK',U)$ and $v \in V$ we have $v \cdot f(K) = f(Kv) \subseteq f(KK') \subseteq U$).

To show that $F(G,A)$ is a topological group, suppose $f_1, f_2 \in F(G,A)$ and $f_1 + f_2 \in N(K,U)$. For each $x \in K$ there are neighborhoods $P_x, Q_x \subseteq A$ of $f_1(x), f_2(x)$, respectively, such that $P_x + Q_x \subseteq U$. Because G is locally compact, there is a compact set $K_x \subseteq f_1^{-1}(P_x) \cap f_2^{-1}(Q_x)$ containing a neighborhood U_x of x. The U_x's cover all of K, so we can choose a finite subcovering. Let $K_1, \dots, K_n$ be the corresponding K_x's, $P_1, \dots, P_n$ the corresponding P_x's, and $Q_1, \dots, Q_n$ the corresponding Q_x's. Let $N_1 = \bigcap_{i=1}^n N(K_i, P_i)$ and $N_2 = \bigcap_{i=1}^n N(K_i, Q_i)$. Then N_k is a neighborhood of f_k $(k = 1,2)$ in $F(G,A)$ such that $N_1 + N_2 \subseteq N(K,U)$ (if $h_1 \in N_1, h_2 \in N_2$ then for each $x \in K$ there is a corresponding $K_i \ni x$, and $h_1(K_i) \subseteq P_i, h_2(K_i) \subseteq Q_i, P_i + Q_i \subseteq U \Rightarrow (h_1 + h_2)(x) \in (h_1 + h_2)(K_i) \subseteq h_1(K_i) + h_2(K_i) \subseteq U$, i.e. $h_1 + h_2 \in N(K,U)$). This shows that the map $+ : F(G,A) \times F(G,A) \to F(G,A)$ is continuous. Finally, the map $- : F(G,A) \to F(G,A)$ is continuous because $-N(K,U) = N(K,-U)$.

The natural map $\phi : A \to F(G, A) : a \mapsto (g \mapsto g \cdot a)$ is continuous: suppose $a \in A$ is in the preimage of $N(K, U)$. Note $\phi^{-1}(N(K, U))$ is the set of $a \in A$ with $K \cdot a \subseteq U$. For each $x \in K$ there are neighborhoods $G_x \subseteq G$ of x and $A_x \subseteq A$ of a such that $G_x \cdot A_x \subseteq U$, and by compactness of K, finitely many such neighborhoods $G_1, \ldots, G_n$ cover K; let $A_1, \ldots, A_n$ be the corresponding open sets in A. Then $A_0 := \bigcap_{i=1}^{n} A_i$ is a neighborhood of a in $\phi^{-1}(N(K, U))$ since each $x \in K$ is contained in some G_i, and $G_i \cdot A_i \subseteq U \Rightarrow G_i \cdot A_0 \subseteq U$. Furthermore, ϕ is proper, i.e. open onto its image: the image of the open set $U \subseteq A$ is $\mathrm{Im}(\phi) \cap N(\{e\}, U)$.

In fact, ϕ has a continuous retraction $f \mapsto f(e)$, which is a homomorphism of abelian groups. This implies $\phi \in S$ because in the category $\mathcal{M}_G^{ne}$ of topological G-modules with continuous (not G-equivariant) homomorphisms (see Section 1.3.3.1), the sequence $0 \to A \to F(G, A) \to F(G, A)/A \to 0$ is split (by Lemma 1.2.4)[1].

Finally, we show that the map $\phi : A \mapsto F(G, A)$ kills the cohomology. Suppose $f \in Z^n(G, A)$, so $\delta_n f = 0$. Define $f' \in C^{n-1}(G, F(G, A))$ by $f'(g_1, \ldots, g_{n-1})(x) = f(x, g_1, \ldots, g_{n-1})$ (f' is easily seen to be continuous). Then, if $\phi_n : C^n(G, A) \to C^n(G, F(G, A))$ is the induced map on cohomology $f \mapsto \phi \circ f$ then $\phi_n(f) = \delta_{n-1} f'$, i.e. $\phi_n(f) = 0 \in H_c^n(G, F(G, A))$: indeed,

$$\phi_n(f)(g_1, \ldots, g_n)(x) = x \cdot f(g_1, \ldots, g_n)$$

whereas

$$\begin{aligned} \delta_{n-1} f'(g_1, \ldots, g_n)(x) &= (g_1 \cdot f'(g_2, \ldots, g_n))(x) - f'(g_1 g_2, g_3, \ldots, g_n)(x) + \cdots \\ &\quad + (-1)^{n-1} f'(g_1, \cdots, g_{n-2}, g_{n-1} g_n)(x) + (-1)^n f'(g_1, \cdots, g_{n-1})(x) \\ &= f(x g_1, g_2, \ldots, g_n) - f(x, g_1 g_2, g_3, \cdots, g_n) \\ &\quad + \cdots + (-1)^{n-1} f(x, g_1, \cdots, g_{n-2}, g_{n-1} g_n) + (-1)^n f(x, g_1, \cdots, g_{n-1}) \end{aligned}$$

Replacing g_0 by x in Equation (3) and setting that equation equal to zero (since f is a cocycle), we see the two are the same.

Thus, by Theorem 1.2.13 we have

[1] This argument is due to Theo Bühler - private communication.

Theorem 3.1.1. $\mathrm{Ext}^n_{\mathcal{M}_G,S}(\mathbb{Z},A) = H^n_c(G,A)$ *for locally compact G.*

Note that $F(G,A)$ is Hausdorff if A is: we just need to show $F(G,A)$ is T_1, i.e. if $f : G \to A$ is a continuous map, we need to show $\{f\}$ is closed in $F(G,A)$. Given $f' \in F(G,A), f' \neq f$, there is some element $g \in G$ with $f'(g) \neq f(g)$, so $N(\{g\}, A \setminus \{f(g)\})$ is a neighborhood of f' that does not contain f (since A is Hausdorff, $\{f(g)\}$ is closed). This shows that $F(G,A)$ is T_1. Thus we also have:

Corollary 3.1.2. $\mathrm{Ext}^n_{\mathcal{M}^H_G,S}(\mathbb{Z},A) = H^n_c(G,A)$ *for locally compact G and Hausdorff A.*

Also note that if G is locally compact and σ-compact and A is a complete metric topological G-module, then $F(G,A)$ is a complete metric topological G-module. Indeed, since A is metrizable, the topology of compact convergence induces the topology on $F(G,A)$. It is well-known that if X is σ-compact and Y is a complete metric space, then the set of functions $X \to Y$, with the compact convergence topology, is a complete metric space (for example, see exercise 46.10 in [**22**]). Finally, if G is locally compact, then G is compactly generated, hence the set of continuous functions $G \to A$ is closed in the set of all functions under the compact convergence topology (for example, see Theorem 46.5 in [**22**]).

Corollary 3.1.3. *Let G be locally compact and σ-compact. Let A be a completely metrizable G-module. Then* $\mathrm{Ext}^n_{\mathcal{M}^{cm}_G,S}(\mathbb{Z},A) = H^n_c(G,A)$.

3.2. Moore's Measurable Cohomology

Let G be a second countable Hausdorff locally compact group (this implies G is Polish). Moore [**21**] defines the measurable cochain theory $(H^n_m(G,A))^\infty_{n=0}$ in $\mathcal{M}^P_G$ and shows that it has all the good properties one could want for a cohomology theory. In fact, it seems the only way to improve this theory would be to have one that works for more general topological groups G and topological G-modules.

Moore defines two a priori different cohomology theories on $\mathcal{M}^P_G$ which turn out to yield isomorphic cohomologies. For $A \in \mathcal{M}^P_G$, let $C^0(G,A) = \underline{C}^0(G,A) = A$. Let $C^n(G,A)$ be the set of measurable functions from G^n to A, and let $\underline{C}^n(G,A)$ be

$C^n(G,A)/\sim$, where $\sim$ is the relation under which two functions are equivalent if and only if they are equal almost everywhere (with respect some Haar measure on G^n). Define the coboundary operator $\delta_n : C^n(G,A) \to C^{n+1}(G,A)$ as usual, by Equation (3), and define $\delta_n : \underline{C}^n(G,A) \to \underline{C}^{n+1}(G,A)$ in the same way. We defined $H^n_m(G,A)$ to be the n-th cohomology group of the complex $(C^n(G,A), \delta_n)$ in Section 2.1.1.1. Now let $H^n(G,A) = H^n_m(G,A)$, and let $\underline{H}^n(G,A)$ be the n-th cohomology group of the complex $(\underline{C}^n(G,A), \delta_n)$. Clearly, $H^0(G,A) = \underline{H}^0(G,A) = A^G$. Moore proves that both the functors $H^n(G,-)$ and $\underline{H}^n(G,-)$ give an exact connected sequence of functors [**21**, Theorem 4]. Moreover, he puts a topology[2] and a G-module structure on $I(A) := \underline{C}^1(G,A)$ and shows that $I(A)$ is a topological G-module such that there is a proper injection $A \to I(A)$ [**21**, Proposition 13] and $H^n(G,I(A)) = \underline{H}^n(G,I(A)) = 0$ [**21**, Proposition 21]. Thus, by the universality of Ext, we have

Theorem 3.2.1. *Let G be second countable, Hausdorff, and locally compact. Then* $\mathrm{Ext}^n_{\mathcal{M}^P_G, P(\mathcal{M}^P_G)}(\mathbb{Z}, A) = H^n_m(G,A) = \underline{H}^n(G,A)$ *for all n and all $A \in \mathcal{M}^P_G$.*

Of course, the first cohomology group $H^1_m(G,A)$ is the set of measurable crossed homomorphisms modulo the principal ones, and it turns out [**15**, p. 8, Remark 1] that any measurable crossed homomorphisms is continuous. Here we show precisely why this is true. Moore stated [**21**, p. 5, Proposition 5(a)] that a measurable homomorphism is continuous, citing Banach's work [**2**] and Kuratowski's [**16**]. This resulted in some confusion since Banach calls a function measurable if it is the (pointwise) limit of a sequence of continuous functions. As Neeb [**23**] noted, if X is connected and Y is discrete, then any continuous function $f : X \to Y$ is constant, hence any limit of such functions is constant, but there are certainly Borel-measurable functions that are not constant (for example, if $X = [0,1]$ and $Y = \{a,b\}$

[2]The topology on $I(A)$ is given by a metric: given a metric ρ on A, first one chooses a finite measure $d\nu$ on G that is equivalent to the Haar measure and then defines the metric by $\bar{\rho}(f_1, f_2) = \int_G \rho(f_1(x), f_2(x)) d\nu(x)$. It is interesting to note that one could use a similar definition for $C^1(G,A)$ instead of $\underline{C}^1(G,A)$ and get a pseudometric instead; then, the Kolmogorov quotient is precisely $\underline{C}^1(G,A)$.

is the discrete space with two points, we can take the function f with $f(0) = a$ and $f(x) = b$ for $x > 0$). As Moore pointed out, however (see [**24**]), it is indeed proven in Kuratowski's work [**16**] that a Borel-measurable homomorphism is continuous. For completeness, here we give a proof similar to that in Kuratowski's work that shows that any Borel-measurable crossed homomorphism is continuous.

We first recall two easy facts [**37**, p. 52]. (1) A map $f : X \to Y$ is continuous if and only if it is continuous at every point $x \in X$, i.e. for every neighborhood N of $f(x)$, $f^{-1}(N)$ is a neighborhood of x. (2) a map $f : X \to Y$ is continuous at $x \in X$ if and only if, for every net $x_\delta \to x$, we have $f(x_\delta) \to f(x)$ (if X is first-countable, it is enough to consider sequences instead of nets).

LEMMA 3.2.2. *Let G be a topological group and A a topological G-module. If $f : G \to A$ is a crossed homomorphism (i.e. $f(xy) = xf(y) + f(x)$) that is continuous at a point x_0, then f is continuous everywhere.*

PROOF. First note that the continuity of the map $G \times A \to A : (g, a) \mapsto g \cdot a$ implies that if $a_\delta \to a \in A$ and $g_\delta \to g \in G$ then $g_\delta \cdot a_\delta \to g \cdot a$; in particular, $g \cdot a_\delta \to g \cdot a$, and similarly when A is replaced by G. Also, the continuity of the maps $A \times A \to A : (a, b) \mapsto a \pm b$ implies that if $a_\delta \to a \in A$ and $b_\delta \to b \in A$ then $a_\delta - b_\delta \to a - b$.

A little algebra with crossed homomorphisms shows that $f(e) = 0$ and $f(g^{-1}) = -g^{-1} \cdot f(g)$. Now suppose we have a convergent net $x_\delta \to x$. Let $x = gx_0$. Then

$$\begin{aligned}
g^{-1}x_\delta &\to x_0 \\
f(g^{-1}x_\delta) &\to f(x_0) \\
(g^{-1} \cdot f(x_\delta) + f(g^{-1})) &\to f(x_0) \\
(g^{-1} \cdot f(x_\delta) - g^{-1} \cdot f(g)) &\to f(x_0) \\
(f(x_\delta) - f(g)) &\to g \cdot f(x_0) \\
f(x_\delta) &\to g \cdot f(x_0) + f(g) = f(gx_0) = f(x)
\end{aligned}$$

so f is continuous at x. □

Definition 3.2.3. *A subset is* **nowhere dense** *if its closure has empty interior. A subset is* **first category** *if it is the countable union of nowhere dense sets. Otherwise, the subset is said to be* **second category**. *Following* [**16**], *a subset B in a topological space X is* **Baire** *if there is an open set $U \subseteq X$ such that the symmetric difference of B and U is first category.*

It is easy to see that the Baire subsets of a space form a σ-algebra[**16**, p. 88] containing the open sets, hence containing the Borel σ-algebra.

Definition 3.2.4. *A map $f : X \to Y$ of topological spaces satisfies the* **Baire property** *if for every open set $U \subseteq Y$, $f^{-1}(U)$ is a Baire set in X.*

Note that, since the Borel measurable sets are Baire, any Borel measurable map has the Baire property.

Lemma 3.2.5. *If Y is second-countable, then $f : X \to Y$ satisfies the Baire property if and only if there is a first category set $P \subseteq X$ such that $f|X \setminus P$ is continuous.*

Proof. This is proven in [**16**, p. 400]. □

Theorem 3.2.6. *If G is a first countable topological group that is second category in itself and A is a second countable topological G-module then any crossed homomorphism $f : G \to A$ satisfying the Baire property (in particular, any measurable crossed homomorphism) is continuous.*

Proof. The proof follows that of [**2**, Theorem 4, p. 23]. Since f satisfies the Baire property, by Lemma 3.2.5 there is a first category set $P \subseteq G$ such that $f|G \setminus P$ is continuous. Suppose $\lim_{n\to\infty} x_n = e \in G$. Then x_nP is a first category set, for each n. Thus

$$S := P \cup \bigcup_n x_nP$$

is a first category set, hence there is an element x of G which is not in S, i.e. such that $x \notin P$ and $x \notin x_nP$, hence $x_n^{-1}x \notin P$, for all n. But f is continuous at $G \setminus P$, so we have

$$\lim_{n\to\infty} f(x_n^{-1}x) = f(x)$$

Now $f(x_n^{-1}x) = x_n^{-1}f(x) + f(x_n^{-1}) = x_n^{-1}(f(x) - f(x_n))$. Thus

$$f(x) = (\lim_{n\to\infty} x_n^{-1}) \cdot (\lim_{n\to\infty}(f(x) - f(x_n))) = f(x) - \lim_{n\to\infty} f(x_n)$$

so $\lim_{n\to\infty} f(x_n) = 0 = f(e)$. This means f is continuous at $e \in G$, hence continuous everywhere by Lemma 3.2.2. □

Note that a group G is second countable in itself if and only if it is a Baire space, i.e. any first category subset of G has empty interior [**37**, Ex. 115, p. 250]. In particular, the theorem applies in Moore's setting where G is locally compact Polish and A is Polish.

3.3. Michael's Selection Theorem

Ernest Michael developed the famous theory of continuous selections, which is applied in Wigner's work [**35**, p. 4]. For a set X, we will denote the power set of X by 2^X. For a topological space X, we will denote the set of all non-empty closed subsets of X by $\mathcal{E}(X)$. For two sets X, Y, a **carrier** $\phi : X \to 2^Y$ (or $X \to \mathcal{S}$ for some subset $\mathcal{S} \subseteq 2^Y$) is just a function on X that takes values in the subsets of Y. Let X and Y be topological spaces from now on. A **selection** (or **continuous selection**) for the carrier $\phi : X \to 2^Y$ is a continuous map $f : X \to Y$ such that $f(x) \in \phi(x)$ for all $x \in X$. A carrier $\phi : X \to 2^Y$ is **lower semi-continuous** (**l.s.c.**) if for every open subset V of Y the set $\{x \in X \mid \phi(x) \cap V \neq \emptyset\}$ is open in X.

In Michael's work, the concept of **dimension** is that of Lebesgue dimension [**20**], i.e. $\dim X \leq n$ iff every finite open covering $\mathcal{U}$ of X has an open refinement $\mathcal{V}$ such that every element $x \in X$ is in at most $n + 1$ elements of $\mathcal{V}$, and $\dim X$ is the minimum n such that $\dim X \leq n$. Also, for a closed subset $A \subseteq X$, we have $\dim_X(X \setminus A) \leq n$ if $\dim C \leq n$ for every $C \subseteq X \setminus A$ which is closed in X.

It is unnecessary for this exposition to know what C^n, LC^n, and equi-LC^n mean, but suffice it to say that if $n = -1$ then any topological space X is C^n and LC^n, and any family $\mathcal{S} \subseteq 2^X$ satisfies equi-LC^n.

Recall that a topological space X is **paracompact** if every open cover $\mathcal{U}$ of X has an open refinement $\mathcal{V}$ that is locally finite, i.e. every $x \in X$ has a neighborhood which intersects only finitely many elements of $\mathcal{V}$.

THEOREM 3.3.1 (Michael, Theorem 1.2 in [**20**]). *Let X be a paracompact space, $A \subseteq X$ closed with $\dim_X(X \setminus A) \leq n+1$, Y a complete metric space, $\mathcal{S} \subseteq \mathcal{E}(Y)$ equi-LC^n, and $\phi : X \to \mathcal{S}$ an l.s.c. carrier. Then every selection for $\phi \mid A$ can be extended to a selection for $\phi \mid U$ for some open $U \supseteq A$. If also every $S \in \mathcal{S}$ is C^n, then one can take $U = X$.*

The following corollary is slightly stronger than Theorem M that Wigner cited from Michael [**35**, p. 4].

COROLLARY 3.3.2 ([**35**]). *If B is a complete metric topological group, C is any T_1 topological space, $\beta : B \to C$ is an open map, X is a zero-dimensional paracompact space, and $q : X \to C$ is a continuous map, then there is a continuous map $\rho : X \to B$ such that $\beta \circ \rho = q$.*

PROOF. Consider the carrier $\phi : X \to \mathcal{E}(B) : x \mapsto \beta^{-1}(q(x))$ (note $\beta^{-1}(q(x))$ is closed since C is T_1). This carrier is l.s.c. because for an open subset $V \subseteq B$ we have

$$\begin{aligned}\{x \in X \mid \phi(x) \cap V \neq \emptyset\} &= \{x \in X \mid \beta^{-1}(q(x)) \cap V \neq \emptyset\} \\ &= \{x \in X \mid \exists v \in V, \beta(v) = q(x)\} \\ &= q^{-1}(\beta(V)),\end{aligned}$$

which is open. In Theorem 3.3.1, take $A = \emptyset$ and $Y = B$. Then $\dim_X(X \setminus A) = 0$ since any closed subset C of a zero-dimensional space X is zero-dimensional (given a finite cover $\mathcal{U}$ of C, pull back the open sets to open sets in X and add $X \setminus C$ to the cover - this is a cover of X, so there is a finite disjoint open refinement, which gives a finite disjoint open refinement of $\mathcal{U}$). The theorem says that there is a continuous function $\rho : X \to B$ such that $\rho(x) \in \beta^{-1}(q(x))$ for all x, i.e. $\beta \circ \rho = q$. □

Recall that a space X is **σ-compact** if it is the union of countably many compact subspaces. Suppose G is a zero-dimensional, locally compact, σ-compact, Hausdorff group. Then we claim G^n is a paracompact zero-dimensional space for all $n \geq 1$. It is easy to see that a finite product of σ-compact/locally compact/zero-dimensional spaces is still σ-compact/locally compact/zero-dimensional. Now we show that a locally compact, σ-compact Hausdorff topological space X is paracompact. Suppose we have a covering $\mathcal{U} = \{U_\alpha\}$ of X. By σ-compactness, $X = \bigcup_{n=1}^{\infty} K_n$, where each K_n is compact. By local compactness and Hausdorffness, each point $x \in X$ has a base consisting of compact neighborhoods, i.e. compact sets C_x containing a neighborhood U_x of x. These U_x cover each K_n, so we can find a finite subcovering $\{U_n^1, \ldots, U_n^{N_n}\}$ for each K_n. Thus we have a countable collection of compact sets $\{\{K_n^i\}_{i=1}^{N_n}\}_{n=1}^{\infty}$ that cover X and without loss of generality we can assume these are the $K_1, K_2, \ldots$ and that there is an open set $U_n \subseteq K_n$ for each n such that the U_n cover all of X. Now, for each n, $\mathcal{U}$ provides a covering of K_n, so we can choose a finite subcollection $U_n^1, \ldots, U_n^{k_n} \in \mathcal{U}$ that cover K_n. We construct a refinement $\mathcal{V}$ of $\mathcal{U}$ as follows: let $V_n^i = U_n^i \setminus (K_1 \cup \cdots \cup K_{n-1})$ (with $V_1^i = U_1^i$) for all $n = 1, 2, \ldots$ and $i = 1, \ldots, k_n$. Note that since X is Hausdorff, K_1 is closed so $U_2^i \setminus K_1$ is open for all $i = 1, \ldots, k_2$ (this is why the group G must be Hausdorff - a local base of compact neighborhoods exists if G is just locally compact and not Hausdorff, but then the compact sets are not necessarily closed). Set $\mathcal{V} = \{V_n^i\}_{i=1}^{k_n}{}_{n=1}^{\infty}$. Then $\mathcal{V}$ is locally finite because each $x \in X$ lies in some U_n, and hence it can only lie in V_m^i for $m \leq n$.

Theorem 3.3.3 ([**35**]). *Suppose G is locally compact, σ-compact, and zero-dimensional. For A in $\mathcal{M}_G^{cm}$ and all n, we have $H_c^n(G, A) = \mathrm{Ext}^n_{\mathcal{M}_G^{cm}, P(\mathcal{M}_G^{cm})}(\mathbb{Z}, A)$.*

Proof. Clearly, $H_c^0(G, A) = \mathrm{Ext}^0_{\mathcal{M}_G^{cm}, P(\mathcal{M}_G^{cm})}(\mathbb{Z}, A)$. By the universality of the $\mathrm{Ext}^i_{\mathcal{M}_G^{cm}, P(\mathcal{M}_G^{cm})}(\mathbb{Z}, A)$, we just need to show that the $H_c^i(G, A)$ are effaceable and give an exact connected sequence of functors. We showed effaceability in Section 3.1.

If we have an exact sequence $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ in $P(\mathcal{M}_G^{cm})$ then we obtain an exact sequence $0 \to C^n(G, A) \to C^n(G, B) \to C^n(G, C) \to 0$ for every n, since if we have a map $q \in C^n(G, C)$ then by Corollary 3.3.2 there is a map $\rho \in C^n(G, B)$ that maps to q. Thus we get an exact sequence on cohomology by the Snake Lemma. □

Corollary 3.3.4. *Suppose G is locally compact, σ-compact, and zero-dimensional. For A in $\mathcal{M}_G^P$ and for all n, we have $H_c^n(G, A) = H_m^n(G, A)$.*

Proof. The proof of the preceding theorem shows that $H_c^n(G, A) = \mathrm{Ext}^n_{\mathcal{M}_G^P, P(\mathcal{M}_G^P)}(\mathbb{Z}, A)$, which is Moore's cohomology by Theorem 3.2.1. □

CHAPTER 4

Ext **for Pseudometric and Complete Metric G-Modules**

In this chapter we prove that for topological G-modules A and B that are complete metric, the groups $\mathrm{Ext}^n(A, B)$ are the same whether we are working in $(\mathcal{M}_G, P(\mathcal{M}_G)), (\mathcal{M}_G^{pm}, P(\mathcal{M}_G^{pm}))$, or $(\mathcal{M}_G^{cm}, P(\mathcal{M}_G^{cm}))$, a result quoted by Wigner in [**35**] from Lawrence Brown, whose proof was never published.

4.1. Background

A pseudometric (see Section 1.3.3.2) d on a group G is called **left-** (resp. **right-**) **invariant** if $d(gx, gy) = d(x, y)$ (resp. $d(xg, yg) = d(x, y)$) for all $g, x, y \in G$. If G is abelian, these notions are equivalent and called **translation-invariant**.

A topological group G is called **weakly separable**[1] if any uniform cover of G has a countable subcover. Recall that an open cover $\{U_i\}$ of G is a **uniform cover** if there exists an open set $U \subseteq G$ such that either (1) $\forall g \in G\ \exists i\ gU \subseteq U_i$ or (2) $\forall g \in G\ \exists i\ Ug \subseteq U_i$, depending on whether we are talking about the left-uniform structure of G or the right-uniform structure. Thus G is weakly separable if and only if for every open set $U \subseteq G$, the cover $\{gU\}_{g \in G}$ has a countable subcover, or equivalently if for every open set $U \subseteq G$ the cover $\{Ug\}_{g \in G}$ has a countable subcover.

Theorem 4.1.1. *If a topological group G is first countable (or equivalently, first countable at $e \in G$), then its topology is induced by a left-invariant pseudometric.*

Proof. This is Theorem 12.2.3 of [**37**]. □

Corollary 4.1.2. *The topology of a pseudometrizable topological group is induced by a left-invariant pseudometric.*

[1] This is the terminology used by Wigner and Lawrence Brown [**35**]; it seems the more modern terminology is **trans-separable**, where "trans" stands for "translation" [**7**]

Proof. The collection $\{B(e, 1/n)\}_{n=1}^{\infty}$ is a local base at $e \in G$. □

Lemma 4.1.3. *Let A be an abelian group and $\mathscr{F}$ a collection of subsets of A such that:*

(a) $\mathscr{F}$ is a filterbase, i.e. $\mathscr{F} \neq \emptyset$, $\emptyset \notin \mathscr{F}$, and for any $U, V \in \mathscr{F}$ there exists $W \in \mathscr{F}$ with $W \subseteq U \cap V$;

(b) each $U \in \mathscr{F}$ is symmetric (i.e. $U = -U$); and

(c) for each $U \in \mathscr{F}$ there exists $V \in \mathscr{F}$ with $V + V \subseteq U$.

Then there is a unique group topology for A such that $\mathscr{F}$ is a local base for the neighborhoods of $0 \in A$.

Proof. This is Theorem 12.1.6 in [**37**]. Note that $0 \in U$ for all $U \in \mathscr{F}$ since there is a symmetric nonempty neighborhood $V \subseteq U$ such that $V + V \subseteq U$. □

Corollary 4.1.4. *Suppose A is a G-module and $\mathscr{F}$ is a collection of subsets of A satisfying the conditions in Lemma 4.1.3 and also the following:*

(d1) for every $W \in \mathscr{F}$ there exists a neighborhood U of $e \in G$, and there exists $V \in \mathscr{F}$ such that $U \cdot V \subseteq W$;

(d2) for every $g \in G$ and every $U \in \mathscr{F}$ there exists $V \in \mathscr{F}$ such that $g \cdot V \subseteq U$; and

(d3) for every $x \in A$ and every $U \in \mathscr{F}$ there exists a neighborhood V of $e \in G$ such that $V \cdot x - x \subseteq U$.

Then there is a unique topology for A such that $\mathscr{F}$ is a local base for the neighborhoods of $0 \in A$ and A is a topological G-module under this topology.

Proof. By Lemma 4.1.3 there is a unique group topology for A, so we just have to show this topology is a topology for a topological G-module. First, (d1) and (d2) imply

(d4) for every $W \in \mathscr{F}$ and for every $g \in G$ there exists a neighborhood U of g in G and there exists $V \in \mathscr{F}$ such that $U \cdot V \subseteq W$.

Indeed, given $W \in \mathscr{F}$ and $g \in G$, there exists $W' \in \mathscr{F}$ with $g \cdot W' \subseteq W$ by (d2) and there exists a neighborhood U of $e \in G$ and there exists $V \in \mathscr{F}$ such that $U \cdot V \subseteq W'$ by (d1), hence $gU \cdot V \subseteq g \cdot W' \subseteq W$.

Second, (d3) and (d4) imply

(d5) for every $x \in A$, $g \in G$, and $U \in \mathscr{F}$ there exists a neighborhood V of $g \in G$ such that $V \cdot x - g \cdot x \subseteq U$.

This is because given $x \in A, g \in G$, and $U \in \mathscr{F}$, there exists a neighborhood W of $g \in G$ and there exists $V \in \mathscr{F}$ such that $W \cdot V \subseteq U$ by (d4), and there exists a neighborhood V' of $e \in G$ such that $V' \cdot x - x \subseteq V$ by (d3), hence $gV' \cdot x - g \cdot x \subseteq g \cdot V \subseteq W \cdot V \subseteq U$.

Finally, to show A is a topological G-module, suppose W_0 is a neighborhood of $g \cdot x$ in A. Then there exists $W \in \mathscr{F}$ such that $W \subseteq W_0 - g \cdot x$ because $\mathscr{F}$ is a local base at 0. Next, there exists $W' \in \mathscr{F}$ with $W' + W' \subseteq W$ by (c). There exists a neighborhood U_1 of $g \in G$ and there exists $V_1 \in \mathscr{F}$ such that $U_1 \cdot V \subseteq W'$ by (d4). There exists a neighborhood U_2 of $g \in G$ such that $U_2 \cdot x - g \cdot x \subseteq W'$ by (d5), i.e. $U_2 \cdot x \subseteq W' + g \cdot x$. Letting $U = U_1 \cap U_2$, we have $U \cdot (V + x) \subseteq U \cdot V + U \cdot x \subseteq W' + W' + g \cdot x \subseteq W + g \cdot x = W_0$. Thus, U is a neighborhood of $g \in G$ and $V + x$ is a neighborhood of x in A such that $U \cdot (V + x) \subseteq W_0$, which means A is a topological G-module. □

4.2. Extensions of Pseudometric G-modules

Lemma 4.2.1. *Suppose G is weakly separable and we are given an injective proper map $\iota : B \to E$ of topological G-modules where B is pseudometrizable. Then there is a topology T' on E which makes E a topological G-module such that*

(1) the induced map $i : B \to (E, T')$ is still an injective proper map,

(2) T' is coarser than the original topology T on E, and

(3) T' is induced by a left-invariant pseudometric.

Proof. We will denote by $B_B(0, \varepsilon)$ the open ε-ball at 0 in B. Construct a local base $\mathscr{F} = \{U_n\}_{n=0}^{\infty}$ for $0 \in E$ as follows. Let $U_0 = E$. For each $n = 1, 2, \dots$ in order, we perform the following steps.

(i) Let V_n^1 be an open set in E such that $V_n^1 \cap B = B_B(0, 1/n)$.

(ii) Let $V_n^2 = V_n^1 \cap U_{n-1}$.

(iii) Let V_n^3 be a neighborhood of $0 \in E$ such that $V_n^3 + V_n^3 + V_n^3 \subseteq V_n^2$ ($0 + 0 + 0 \in V_n'$ so there are open sets G_1, G_2, G_3 containing 0 such that $G_1 + G_2 + G_3 \subseteq V_n'$; let $U_n' = G_1 \cap G_2 \cap G_3$).

(iv) Let V_n^4 be a neighborhood of $0 \in E$ and W_n a neighborhood of $e \in G$ such that $W_n \cdot V_n^4 \subseteq V_n^3$ (such V_n^4, W_n exist since $e \cdot 0 \in V_n^3$); note that $V_n^4 \subseteq V_n^3$.

(v) Since G is weakly separable, the covering $\{W_n g\}_{g \in G}$ has a countable subcovering $\{W_n g_i^{(n)}\}_{i=1}^{\infty}$. Let $V_{n,i} = (g_i^{(n)})^{-1} \cdot V_n^4$, and let $V_n^5 = V_n^4 \cap \bigcap_{m=1}^{n} \bigcap_{i=1}^{n} V_{m,i}$.

(vi) Let $U_n = V_n^5 \cap -V_n^5$.

We check that $\mathscr{F}$ satisfies the requirements of Corollary 4.1.4. Clearly $\mathscr{F}$ is nonempty and $\varnothing \notin \mathscr{F}$ since each U_n contains 0. It is trivial to check these requirements for U_0 (for all $n \geq 1$ we have $U_n \subseteq U_0$), so we check the requirements are satisfied for $U_n, n \geq 1$. For any $m \leq n, U_n \subseteq U_m \cap U_n$ by step (ii), so property (a) of Lemma 4.1.3 is satisfied. By step (vi), each U_n is symmetric, so (b) is satisfied. By step (iii), $U_{n+1} + U_{n+1} \subseteq U_n$, so (c) is satisfied. Next, (d1) is satisfied because of step (iv) in the construction above. To check (d2), let $g \in G$ and U_n be given; then $g \in W_{n+1} g_i^{(n+1)}$ for some i. Let $N = \max\{n + 1, i\}$. Then $U_N \subseteq V_{n+1,i} = (g_i^{(n+1)})^{-1} \cdot V_{n+1}^4$ (if $i \leq n+1$ then $U_N = U_{n+1} \subseteq V_{n+1}^5 \subseteq V_{n+1,i}$; otherwise, $U_N = U_i \subseteq V_i^5 \subseteq V_{n+1,i}$). Thus $g \cdot U_N \subseteq W_{n+1} g_i^{(n+1)} \cdot ((g_i^{(n+1)})^{-1} \cdot V_{n+1}^4) = W_{n+1} \cdot V_{n+1}^4 \subseteq V_{n+1}^3 \subseteq U_n$. The condition (d3) is automatically satisfied since each U_n is open in T and E is a topological G-module under T. Thus, by Corollary 4.1.4, T' is a topology for a topological G-module.

Now we check (1) the induced map $i : B \to (E, T')$ is a homeomorphism onto its image. To show that i is continuous, it is enough to show it is continuous at 0, which is true since any neighborhood of 0 in (E, T') contains U_n for some n and $U_n \cap B = \iota^{-1}(U_n)$ is open. To show that i is open onto its image, note that for any open set $U \subseteq B$ and any $x \in U$ there exists n with $B_B(x, 1/n) \subseteq U$, and $(U_n + i(x)) \cap B$ is a neighborhood of $i(x)$ contained in $i(U)$. Hence $i(U)$ is covered by open subsets of $i(B)$ and therefore is open in $i(B)$. (2) T' is coarser than T: let U be an open set in T' and $x \in U$. Then there exists n such that $x + U_n \subseteq U$, and $x + U_n$ is open in T, so

U is covered by open sets in T, which implies U is open in T. (3) T' is induced by a left-invariant pseudometric by Theorem 4.1.1. □

Remark. Note that it would be enough to let V_n^3 in step (iii) above be a neighborhood of 0 such that $V_n^3 + V_n^3 \subseteq V_n^2$, but the proof of Theorem 4.1.1 involves taking a local base $\{U_n\}_{n=1}^\infty$ at 0 such that $U_n + U_n + U_n \subseteq U_{n-1}$ for all n; then we know explicitly that under the pseudometric constructed from $\{U_n\}_{n=1}^\infty$, if $x \in U_n$ then $d(0, x) \leq 2^{-n}$. This implies that if $(i(b_n))$ is a Cauchy sequence of elements in E then (b_n) is a Cauchy sequence in B. For every $\varepsilon > 0$, if n is a natural number with $1/n < \varepsilon$, we can find a natural number N such that for all $p, q \geq N$ we have $d_E(i(b_p), i(b_q)) < 2^{-n}$. This means $i(b_p - b_q) \in U_n$, so $d_B(b_p, b_q) < 1/n < \varepsilon$, i.e. (b_n) is Cauchy in B.

Lemma 4.2.2. *Let G be a weakly separable topological group. Let A and B be pseudometrizable topological G-modules. The natural map $\mathrm{Ext}^1_{\mathcal{M}_G^{pm}, P(\mathcal{M}_G^{pm})}(A, B) \to \mathrm{Ext}^1_{\mathcal{M}_G, P(\mathcal{M}_G)}(A, B)$ is an isomorphism. In other words, for any extension*

$$X : 0 \to B \xrightarrow{i} E \xrightarrow{q} A \to 0$$

where A and B are pseudometrizable, E is pseudometrizable as well.

Proof. Given an extension X as above of topological G-modules with A and B pseudometric, let T_B be the topology induced on E from B by Lemma 4.2.1 and let T_A be the weak topology on E with respect to q. Now T_A is a pseudometric topology making E a topological G-module (see Lemma 1.3.13); the pseudometric is given by $d(x, y) = d(q(x), q(y))$. Let T' be the sup of T_A and T_B (see Section 1.3.3.3). It is easy to see that the sup of any collection of topologies under which E is a topological G-module also makes E a topological G-module. Since both T_A and T_B are pseudometric and coarser than the original topology T on E, the sup topology T' is a pseudometric topology coarser than T (if d_1, d_2 are pseudometrics on a set, we can define the pseudometric d by $d(a, b) = d_1(a, b) + d_2(a, b)$ and this will give the sup topology of the two pseudometric topologies corresponding to d_1 and d_2).

Now it is easy to see that we have a map of short exact sequences

$$\begin{array}{ccccccccc} 0 & \to & B & \stackrel{i}{\to} & E & \stackrel{q}{\to} & A & \to & 0 \\ & & \| & & \downarrow & & \| & & \\ 0 & \to & B & \stackrel{i'}{\to} & E' & \stackrel{q'}{\to} & A & \to & 0 \end{array}$$

where E' is E with the topology T'. Indeed, the only thing to check is that the bottom row is exact. The induced map $i' : B \to E'$ is continuous since the preimage of every set in T_A is B or $\emptyset$, and i' is open as a map onto its image because T' is finer than T_B. The induced map $q' : E' \to A$ is continuous because T' is finer than T_A, and q' is open because T' is coarser than the original topology on E. Thus, by the Short Five Lemma (1.2.10) we have $E \cong E'$ and the lemma follows. □

Theorem 4.2.3. *Let G be a weakly separable topological group. Let A and B be pseudometrizable topological G-modules. The natural map*

$$\mathrm{Ext}^n_{\mathcal{M}_G^{pm},P(\mathcal{M}_G^{pm})}(A,B) \to \mathrm{Ext}^n_{\mathcal{M}_G,P(\mathcal{M}_G)}(A,B)$$

is an isomorphism for all $n \geq 0$.

Proof. We will use Theorem 1.2.11; clearly the inclusion $\mathcal{M}_G^{pm} \hookrightarrow \mathcal{M}_G$ is fully faithful, exact, and additive. The surjectivity for $n = 1$ follows from Lemma 4.2.2. We just need to show that the hypothesis (E) of Theorem 1.2.11 holds. If we have a proper monomorphism $m : B \hookrightarrow E$ in $\mathcal{M}_G$ (resp. $\mathcal{M}_G^H$), where B is pseudometrizable then, using Lemma 4.2.1, let E' be E with the topology induced from B. Then E' is a pseudometrizable G-module, and the identity map $f : E \to E'$ clearly commutes with the inclusions $B \hookrightarrow E$ and $B \hookrightarrow E'$. □

4.3. Extensions of Complete Metric G-modules

Lemma 4.3.1. *Let A be a metrizable topological G-module. Then for any translation-invariant metric d on A (which always exists by Corollary 4.1.2), the metric completion B of (A, d) is a topological G-module.*

Proof. It is well-known that B is the space of equivalence classes of Cauchy sequences $(x_n) \subseteq A$, where (x_n) and (y_n) are equivalent if $\lim_{n\to\infty} d(x_n, y_n) = 0$. B has the metric $d'((x_n), (y_n)) = \lim_{n\to\infty} d(x_n, y_n)$. It is easy to see that B is a topological abelian group, i.e. the negation operation $-(x_n) = (-x_n)$ and the addition operation $(x_n) + (y_n) = (x_n + y_n)$ are well-defined and continuous.

We show that the action $G \times B \to B : (g, (x_n)) \mapsto (g \cdot x_n)$, is well-defined and continuous. Suppose (x_n) and (y_n) are two Cauchy sequences representing the same element in B and that we are given $\varepsilon > 0$. By the continuity of the map $G \times A \to A$, since $g \cdot 0 = 0$, there exists $\delta > 0$ such that $d(x, 0) < \delta \Rightarrow d(g \cdot x, 0) < \varepsilon/2$. There exists N such that for all $n \geq N$ we have $d(x_n, y_n) < \delta$, and then $d(g \cdot x_n, g \cdot y_n) = d(g \cdot (x_n - y_n), 0) < \varepsilon/2$ for all $n \geq N$. But this means $d((g \cdot x_n), (g \cdot y_n)) \leq \varepsilon/2 < \varepsilon$, hence the sequences $(g \cdot x_n)$ and $(g \cdot y_n)$ represent the same element in B.

To show that the map $G \times B \to B$ is continuous, let $g \in G$, (x_n) be a Cauchy sequence, and $\varepsilon > 0$. Then (by continuity of $G \times A \to A$) there exists a neighborhood V_1 of g in G and $\delta > 0$ such that $h \in V_1, d(x, 0) < 3\delta \Rightarrow d(h \cdot x, 0) < \varepsilon/3$. There exists N such that $d(x_N, x_n) < \delta$ for all $n \geq N$. By continuity of $G \times A \to A$, there is a neighborhood V' of g such that for all $h \in V_2$ we have $d(h \cdot x_N, g \cdot x_N) < \varepsilon/3$. We claim that for any $h \in V = V_1 \cap V_2$ and any Cauchy sequence (y_n) with $d'((x_n), (y_n)) < \delta$ we have $d'((h \cdot y_n), (g \cdot x_n)) < \varepsilon$. First, $d(x_N - x_n, 0) < \delta < 3\delta$, so $d(h \cdot x_N, h \cdot x_n) < \varepsilon/3$ for all $n \geq N$, hence

$$d'((h \cdot x_N), (h \cdot x_n)) \leq \varepsilon/3$$

Also, there exists M such that $d(y_p, y_q) < \delta$ for all $p, q \geq M$. Then for any fixed $m \geq M$ we have

$$\begin{aligned} d(x_N, y_m) &= d'((x_N), (y_m)) \\ &\leq d'((x_N), (x_n)) + d'((x_n), (y_n)) + d'((y_n), (y_m)) \\ &< \delta + \delta + \delta = 3\delta \end{aligned}$$

Therefore, $d(g \cdot x_N, g \cdot y_m) < \varepsilon/3$ for all $m \geq M$, and

$$d'((g \cdot x_N), (g \cdot y_n)) \leq \varepsilon/3$$

In total, $d'((h \cdot x_n), (g \cdot y_n)) \leq d'((h \cdot x_n), (h \cdot x_N)) + d'((h \cdot x_N), (g \cdot x_N)) + d'((g \cdot x_N), (g \cdot y_n)) < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$. □

Lemma 4.3.2. *Let G be a weakly separable topological group. Suppose we have an exact sequence of topological G-modules* $0 \to B \xrightarrow{\beta} E \xrightarrow{\alpha} A \to 0$. *If A and B both have any of the following properties, so does E: (1) Hausdorff, (2) pseudometrizable, (3) metrizable, (4) completely metrizable.*

Proof. If A is Hausdorff, then the image of B must be closed in E. If B is Hausdorff, then $\{0\}$ is closed in B, hence $\{0\}$ is closed in E, i.e. E is Hausdorff (this is true with no assumptions on G). If A and B are pseudometrizable, the proof of Theorem 4.2.3 shows that E is, too. Thus, if A and B are metric G-modules, so is E.

Now suppose A and B are completely metrizable. By the above, E is a metric G-module, with metric $d = d'_B + d_A$, where d'_B is induced from the metric d_B on B as in the proof of Theorem 4.2.3 and d_A is induced from the metric on A. (Note that d'_B restricted to B is not d_B, but equivalent to d_B, i.e. they induce the same topology on B.) We just have to show E is complete, so let $\bar{E}$ be the completion of E with respect to d. By the universal property of completion, since A is complete we get a continuous homomorphism of topological abelian groups $\bar{\alpha} : \bar{E} \to A$. The embedding of E as a dense subset of $\bar{E}$ induces an embedding of B in the kernel K of $\bar{\alpha}$:

$$\begin{array}{ccccccccc} 0 & \to & B & \xrightarrow{\beta} & E & \xrightarrow{\alpha} & A & \to & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & K & \to & \bar{E} & \xrightarrow{\bar{\alpha}} & A & \to & 0 \end{array}$$

Just as in [**21**, Proposition 3], we can show that the image of B in K is dense. Suppose $x \in K$; then there is a sequence (x_n) of elements of E with $x_n \to x$. Since $\alpha(x_n) = \bar{\alpha}(x_n) \to 0$, for every $k = 1, 2, \ldots$ there is a natural number N_k such that for all $n \geq N_k$ we have $d(x_n + B, 0 + B) < 1/k$, so there exists $b_k \in B$ with $d(x_{N_k}, b) < 1/k$.

Then $b_k - x_{N_k} \to 0$ and $b_k = (b_k - x_{N_k}) + x_{N_k} \to 0 + x = x$. Since $b_k \to x$, (b_k) is Cauchy in the metric on E. But by the reasoning in the remark following Theorem 4.2.3, this means (b_k) is Cauchy in B. Since B is complete (under the metric d_B on B) and $b_n \to x$, we have $x \in B$. Therefore, the inclusion $B \hookrightarrow K$ is onto and by diagram chasing, the inclusion $E \hookrightarrow \bar{E}$ is onto, i.e. E is complete. □

Theorem 4.3.3. *Let A and B be complete metric G-modules and G weakly separable. Then the natural map*

$$\mathrm{Ext}^n_{\mathcal{M}_G^{cm}, P(\mathcal{M}_G^{cm})}(A, B) \to \mathrm{Ext}^n_{\mathcal{M}_G, P(\mathcal{M}_G)}(A, B)$$

is an isomorphism for all $n \geq 0$.

Proof. We use Theorem 1.2.11. Again, the inclusion $\alpha : \mathcal{M}_G^{cm} \hookrightarrow \mathcal{M}_G$ is obviously a fully faithful exact additive functor. Surjectivity for $n = 1$ is shown by Lemma 4.3.2, so we just have to show that for any proper monomorphism $i : B \to E$ with B complete metric there is a complete metric G-module E', a proper monomorphism $i' : B \to E'$, and a map $f : E \to E'$ with $f \circ i = i'$. Using Lemma 4.2.1, let E_1 be E with the pseudometric topology induced from B and let $i_1 : B \to E_1$ be the inclusion (of course, setwise $i_1 = i$). Let E_1^0 be the closure of $\{0\}$ in E_1. Then E_1^0 is the intersection of all open sets in E_1 (because E_1 is a topological group), so $E_1^0 = \{x \in E_1 \mid d(x, 0) = 0\}$. Let $E_1^H = E_1/E_1^0$, let $p : E_1 \to E_1^H$ be the quotient map, and let $j = p \circ i_1$. By Lemma 1.3.9, E_1^H has a metric d' given by $d'(p(x), p(y)) = d(x, y)$.

First we show *$j : B \hookrightarrow E_1^H$ is injective.* If $b \in B$ and $i_1(b) \in E_1^0$ then $i_1(b) \in U$ for each open set U in E_1, so $i_1(b) \in i_1(V)$ for each open set V in B (since i is a homeomorphism onto its image). Since i_1 is injective, we have $i_1(\bigcap V) = \bigcap i_1(V)$, so $i_1(b) \in i_1(\bigcap(V))$ (where we take the intersection over all V open in B). But B is Hausdorff, so $\bigcap V = \{0\}$, and $i_1(b) = 0 \Rightarrow b = 0$.

Second, *$j : B \hookrightarrow E_1^H$ is a homeomorphism onto its image.* Of course, j is continuous, being the composition of continuous functions. Let V be open in B and $i_1(V) = U \cap i_1(B)$, for some open set U in E_1. Let $x \in i_1(V)$. Then there exists $\varepsilon > 0$ such that $B_{E_1}(x, \varepsilon) \subseteq U$, and $B_{E_1^H}(p(x), \varepsilon) \cap j(B) \subseteq j(V)$ (if $y + E_1^0 \in B_{E_1^H}(p(x), \varepsilon) \cap j(B)$ then we may assume without loss of generality that $y = i_1(b)$ for some $b \in B$, and $d(p(x), y + E_1^0) < \varepsilon \Rightarrow d(x, y) < \varepsilon \Rightarrow y \in B_{E_1}(x, \varepsilon) \cap i_1(B) \subseteq U \cap i_1(B) = i_1(V)$, hence $y + E_1^0 \in j(V)$), which implies j is proper.

Third, *the image $j(B)$ is complete under the metric on E_1^H* (hence closed in E_1^H). Suppose $(j(x_n))$ is a Cauchy sequence in E_1^H with each $x_n \in B$, so $(i_1(x_n))$ is a Cauchy sequence in E_1. By the remark following Lemma 4.2.1, (x_n) is Cauchy in B. Since B is complete, (x_n) converges to some $x \in B$. Since j is continuous, $j(x_n) \to j(x)$. And because E_1^H is Hausdorff, x is the unique point to which $j(x_n)$ converges, so $j(B)$ is complete and closed in E_1^H.

Now let E' be the metric completion of E_1^H. By Lemma 4.3.1, E' is a complete metric topological G-module. Consider the inclusion $i' : B \hookrightarrow E'$ which is the composition $\phi \circ j$, where $\phi : E_1^H \hookrightarrow E'$ is the canonical inclusion. The image $i'(B)$ is a closed subgroup of E' since $j(B)$ is complete in E_1^H. Since ϕ is a proper monomorphism (for any $x \in E_1^H$ and $\varepsilon > 0$, $B_{E'}(\phi(x), \varepsilon) \cap \phi(E_1^H) = \phi(B_{E_1^H}(x, \varepsilon))$) and the composition of two proper monomorphisms is proper, i' is a proper monomorphism. Clearly the induced map $f : E \to E'$ commutes with i and i'. □

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